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Homological mirror symmetry on noncommutative two-tori

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Abstract

Homological mirror symmetry is a conjecture that a category constructed in the A-model and a category constructed in the B-model are equivalent in some sense. We construct a cyclic differential graded (DG) category of holomorphic vector bundles on noncommutative two-tori as a category in the B-model side. We define the corresponding Fukaya's category in the A-model side, and prove the equivalence of the two categories at the level of cyclic categories. We further write down explicitly Feynman rules for higher Massey products derived from the cyclic DG category. As a background of these arguments, a physical explanation of the mirror symmetry for noncommutative two-tori is also given.

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1 Introduction

String theories have provided us with various 'stringy' deformations of the geometries of their target spaces. In general closed strings include gravitons, which induce the deformation of the metric on the target space. Open strings include gauge fields, which define the connections on vector bundles (D-branes). Thus, closed string physics is related directly to the geometry of the target space M, whereas open string physics in general extracts some informations of the geometry in a more abstract algebraic way. Various string dualities are then interpreted as some equivalences between the free energy, *i.e.* , a collection of correlation functions, of string theories. One of such string dualities is mirror symmetry, a symmetry between Calabi-Yau manifolds, which can be interpreted by topological closed string physics. There are two types of topological strings whose target spaces are Calabi-Yau manifolds. One is called the A-model, which depends on the complexified symplectic structure and is independent of the complex structure of the Calabi-Yau manifold. Another one, the B-model in contrast depends on the complex structure only. For a given Calabi-Yau manifold M, the mirror symmetry conjecture claims the existence of a mirror Calabi-Yau manifold $\hat{M}$ such that the A-model closed string on M is equivalent to the B-model closed string on M and vice versa [63]. Homological mirror symmetry conjecture proposed by Kontsevich [35] is thought of as an open string version [64] of this mirror symmetry conjecture. Open string theory in general includes some kind of D-branes. It forms a "D-brane category" (see [39]); D-branes and open strings are identified with objects and morphisms between the objects, respectively, and the free energy determines the composition rules of the morphisms. For the tree open string A-model, the corresponding category is Fukaya's A_∞-category [16]. On the other hand, what is constructed on the B-model side is a category of holomorphic vector bundles or coherent sheaves more generally. The homological mirror symmetry conjecture then states that the Fukaya category on a Calabi-Yau manifold M is in some sense equivalent to the category of coherent sheaves on the mirror dual

Calabi-Yau manifold $\hat{M}$. Now, the conjecture is checked successfully in the case M is an elliptic curve [35, 48, 45, 46], an abelian variety [17, 36], a quartic surface [57], and so on.

Since open string physics describe the target space geometry algebraically, it allows us to consider more extended geometry than the classical ones just as noncommutative geometry [8] does. Some sort of noncommutative geometry are in fact interpreted by open string theories [9, 15, 32, 7, 53, 56, 6]. In the spirit of noncommutative geometry, a noncommutative algebra is regarded as the space of functions on a 'noncommutative space'. Projective modules over the noncommutative algebra are then vector bundles, *i.e.* D-branes over the noncommutative space. Thus, one can consider a D-brane category over the noncommutative space. K-theory is then defined in the framework of C^*-algebra, where the K_0-group consists of the projective modules. Since the Bott periodicity holds as in the case of usual commutative spaces, one can say that a noncommutative C^*-algebra describes some kind of space. Here, although the corresponding open string theory is different from the topological strings in the mirror symmetry set-up, if for instance a complex structure would be defined on some kind of noncommutative Calabi-Yau manifold, one may consider a category of B-model on it.

For such a direction, an ideal candidate is that given by A. Schwarz [55, 12], where a complex structure is introduced on even dimensional noncommutative tori and a holomorphic structure is defined on projective modules over it. In fact for noncommutative (real) two-tori, we succeeded to find the corresponding A-model side and observe a part of the homological mirror symmetry in the previous paper [28]. For a recent discussion see also [33]. The criterion to find the A-model side [28] is the T-duality in string theory. Now, noncommutative tori are interpreted from the physics of open strings [9, 15, 32, 7, 56], and in particular in [28] we explained that two different representations of noncommutative two-tori are related by T-duality [28]. It is known that the T-duality coincides with the mirror symmetry for two-tori [13]. Introducing a complex and holomorphic structure on one side and transforming it by the T-duality we can expect what kind of category should be taken for the A-model side [28]. In this two-dimensional case, the A-model side is essentially independent of the noncommutativity $\theta \in \mathbb{R}$ and the same as the commutative case in [48]. This means the B-model side, the category of holomorphic vector bundles over a noncommutative two-torus, should be essentially independent of θ. Such thought leads to show that the derived categories of holomorphic vector bundles over noncommutative two-tori with different θ are equivalent [47]. There a relation between the noncommutativity θ and the stability condition introduced by Douglas, Bridgeland [14, 5] is also discussed.

This paper is a continuation of our previous work [28]. We construct a cyclic differential graded (DG) category of holomorphic vector bundles on noncommutative two-tori as a category of the B-model side. We clarify the meaning of the mirror symmetry for noncommutative two-tori, especially the relation of the noncommutativity θ with the t-structure, and set up the homological mirror symmetry on noncommutative two-tori. As a part of it, we prove the equivalence of the cyclic DG category to the corresponding category in A-model side as cyclic categories. We further write down explicitly Feynman rules for higher Massey A_∞-products derived from the cyclic DG category.

One may take the zero noncommutativity limit in these results. They then reproduce various results in the commutative case (see [35, 48, 45, 46]) in a different description. One of the advan-

tage of employing these noncommutative tools there is the explicitness. They provide us with a manifestly $SL(2,\mathbb{Z})$ covariant description, where $SL(2,\mathbb{Z})$ is the symmetry of noncommutative tori known as the Morita equivalence (see [49, 8, 34]), and further morphisms of the cyclic DG category can be described explicitly by Hermite polynomials in general. An alternate feature of our approach is that we treat carefully the cyclic structure which the categories have.

In section 2 we reformulate some tools for noncommutative two-tori and differential geometry on it. In subsection 2.1, we recall the definitions of noncommutative two-tori and Heisenberg modules, which are projective modules over noncommutative tori equipped with constant curvature connections. The space of homomorphisms between two Heisenberg modules is also identified with a Heisenberg module. As the composition of the homomorphisms, the tensor product between two Heisenberg modules are given in subsection 2.2. In subsection 2.3 we explain the underlying physics of open string theory. From the viewpoint of *mirror symmetry for noncommutative two-tori*, the meaning of the noncommutativity is clarified. Although the reader can skip this subsection mathematically, it gives a criterion to set up homological mirror symmetry conjecture on noncommutative two-tori. In section 3 we discuss a complex geometry of noncommutative two-tori, its dual symplectic side, and the correspondence between them, the homological mirror symmetry. In subsection 3.1 we recall the holomorphic structures on Heisenberg modules introduced by A. Schwarz [55, 12] and construct a cyclic differential graded (DG) category of (stable) holomorphic vector bundles over a noncommutative two-torus partially based on the work by Polishchuk-Schwarz[47]. In subsection 3.2 we propose a candidate of the corresponding category on the A-model side, the Fukaya's A_∞-category for noncommutative two-tori, and set up a homological mirror symmetry conjecture on noncommutative two-tori. As a part of the conjecture, we prove a categorical mirror symmetry in a stronger sense than the Polishchuk-Zaslow's result for (commutative) elliptic curve [48]. Finally in subsection 3.3 we complete the Feynman rule to generate higher Massey A_∞-products, which form a minimal cyclic A_∞-category, from the cyclic DG category of holomorphic vector bundles on noncommutative two-tori. In this paper, we treat any (graded) vector space over field $k=\mathbb{C}$.

2 Noncommutative two-tori

In this section we recall and arrange some basic tools for noncommutative two-tori to apply them to a mirror symmetry set-up. The definitions of noncommutative two-tori and Heisenberg modules are recalled in subsection 2.1, and the tensor products between two Heisenberg modules are given in subsection 2.2. In subsection 2.3 we explain the underlying open string theory physics, which gives a criterion to set up homological mirror symmetry conjecture on noncommutative two-tori. In this paper, we define noncommutative tori and various structures on them in a different notations from a conventional one in order for them to fit the notation of the mirror symmetry set-up.

2.1 Noncommutative two-tori and Heisenberg modules

A noncommutative two-torus $\mathcal{A}_\theta$ is defined by two unitary generators U_1, U_2 with relation

$$U_1 U_2 = e^{-2\pi i\theta} U_2 U_1 \ . \tag{1}$$

Any element $a \in \mathcal{A}_\theta$ is represented as

$$a = \sum_{(n_1,n_2)\in\mathbb{Z}^2} a_{n_1 n_2} (U_1)^{n_1} (U_2)^{n_2} \ , \quad a_{n_1 n_2} \in \mathbb{C} \ ,$$

where a belongs to the Schwartz space $\mathcal{S}(\mathbb{Z}^2)$. The trace is defined by

$$\mathrm{Tr}(a) = a_{00} \ ,$$

which is just the integral on $\mathbb{T}^2$ when $\theta = 0$. In this paper, we let $\theta \in \mathbb{R}$ irrational for simplicity. But the results of this paper reduce to the ordinary commutative two-tori case by taking the limit $\theta \to 0$, and the rational θ cases are in fact equivalent to the $\theta = 0$ case. A representation of this noncommutative two-torus is the rotation algebras (see [8]). Consider the space of complex-valued continuous functions $C(S^1)$ on S^1 parametrized by $x \in \mathbb{R}$ with periodicity $x \sim x+1$. $U_1, U_2 : C(S^1) \to C(S^1)$ are defined by

$$U_1 a(x) = e^{2\pi i x} a(x) \ , \qquad U_2 a(x) = a(x+\theta) \ .$$

The two generators U_1, U_2 generates the crossed product algebra, which is a noncommutative two-torus $\mathcal{A}_\theta$ above.

In noncommutative geometry, the analogue of vector bundles are projective modules. On noncommutative tori, projective modules, called Heisenberg modules, are given explicitly (see [34]). For each $g = \left(\begin{smallmatrix} q & s \\ p & r \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$, if $p = 0$ we just take $C(S^1)$ above as a module. If $p \neq 0$, a Heisenberg module $E_{g,\theta}$ over $\mathcal{A}_\theta$ is given by the space of $|p|$ copies of functions on $\mathbb{R}$, or more precisely, the Schwartz space $\mathcal{S}(\mathbb{R}\times\mathbb{Z}_{|p|})$. On $f \in E_{g,\theta}$, the action of $\mathcal{A}_\theta$ is defined by

$$\begin{aligned} (U_1 f)(x,j) &= f(x,j) e^{2\pi i (x + j\frac{q}{p})} \\ (U_2 f)(x,j) &= f(x + \frac{q}{p} + \theta, j-1) \ , \end{aligned} \tag{2}$$

where $x \in \mathbb{R}$ and $j = 0, \cdots, p-1$. One can see that U_1 and U_2 in fact satisfy the noncommutativity relation (1). In general, for a projective module E over a C^*-algebra $\mathcal{A}$, the space of endomorphisms of E that commute with the action of $\mathcal{A}$, denoted by $\mathrm{End}_{\mathcal{A}} E$, forms an algebra, and it is called Morita equivalent to $\mathcal{A}$. Such a projective module E is then called a Morita equivalence bimodule. In the case of noncommutative tori, the endomorphism algebra $\mathrm{End}_{\mathcal{A}_\theta} E_{g,\theta}$ is also generated by two generators Z_1, Z_2. It is given as right actions by

$$\begin{aligned} (f Z_1)(x,j) &= f(x,j) e^{2\pi i (\frac{x}{q+p\theta} + \frac{j}{p})} \\ (f Z_2)(x,j) &= f(x + \frac{1}{p}, j - r) \ . \end{aligned} \tag{3}$$

These generators satisfy the following relation

$$Z_1 Z_2 = e^{-2\pi i(g\theta)} Z_2 Z_1 \ , \tag{4}$$

where $g\theta$ is defined by

$$g\theta = \frac{r\theta + s}{q + p\theta} \ . \tag{5}$$

Thus, the endomorphism algebra $\mathrm{End}_{\mathcal{A}_\theta} E_{g,\theta}$ generated by Z_1 and Z_2 again forms a noncommutative torus $\mathcal{A}_{g\theta}$, and $E_{g,\theta}$ is a Morita equivalence $\mathcal{A}_\theta$-$\mathcal{A}_{g\theta}$ bimodule. It is used to show explicitly that $\mathcal{A}_\theta$ and $\mathcal{A}_{g\theta}$ are Morita equivalent (see[51, 52, 34]). It is known that any finitely generated projective module over $\mathcal{A}_\theta$ is isomorphic to the direct sum of Heisenberg modules $E_{g,\theta}$ with $g \in SL(2,\mathbb{Z})$ (see[10, 34]). In such a sense $E_{g,\theta}$ for each $g \in SL(2,\mathbb{Z})$ is called a *basic module* in [47]. This fact implies that it is enough to concentrate our arguments to these basic modules.

A connection $\nabla_i : E_{g,\theta} \to E_{g,\theta}$, $i = 1, 2$ is defined so that

$$\nabla_i(U_j f) = (2\pi i \delta_{ij} U_j) f + U_j(\nabla_i f)$$

holds. Here the first term of the right hand side means that ∇_i acts to U_j as a derivation for i-direction. It is known that $E_{g,\theta}$ can be equipped the following constant curvature connection.

$$\nabla_1 = \frac{\partial}{\partial x} - \frac{2\pi i\beta}{q + p\theta} \ , \quad \nabla_2 = -\frac{2\pi i p}{q + p\theta} x - \frac{2\pi i \alpha}{q + p\theta} \ , \qquad \alpha, \beta \in \mathbb{R} \ . \tag{6}$$

Thus, modules $E_{g,\theta}$ has a constant curvature $[\nabla_1, \nabla_2] = -\frac{2\pi i p}{q+p\theta}$. In eq.(6), α and β parametrize the moduli of constant curvature connection [10, 34]. By gauge transformation $\nabla_i \to (Z_j)^{-1} \nabla_i Z_j$, we have $\alpha \sim \alpha + 1$ and $\beta \sim \beta + 1$. [1]

Next, we would like to consider the space of homomorphisms $\mathrm{Hom}(E_{g_a,\theta}, E_{g_b,\theta})$ between two projective modules $E_{g_a,\theta}$ and $E_{g_b,\theta}$, where $g_a, g_b \in SL(2,\mathbb{Z})$. In fact, $\mathrm{Hom}(E_{g_a,\theta}, E_{g_b,\theta})$ can be identified with an $\mathcal{A}_{\theta_a}$-$\mathcal{A}_{\theta_b}$ bimodule, where $\theta_a := g_a\theta$. If $g_a = g_b$, it is just $\mathcal{A}_{\theta_a}$, whereas if $g_a \neq g_b$, it is the Heisenberg module E_{g_{ab},θ_a}, where we introduced the following notation

$$g_{ab} = g_a^{-1} g_b = \begin{pmatrix} r_a & -s_a \\ -p_a & q_a \end{pmatrix} \begin{pmatrix} q_b & s_b \\ p_b & r_b \end{pmatrix} = \begin{pmatrix} q_{ab} & s_{ab} \\ p_{ab} & r_{ab} \end{pmatrix} \ . \tag{7}$$

There exists a natural identification between $\mathcal{A}_{\theta_a}$-$\mathcal{A}_{\theta_b}$ bimodule and $\mathcal{A}_{\theta_b}$-$\mathcal{A}_{\theta_a}$ bimodule. Namely, one can construct a canonical map $E_{g_{ab},\theta_a} \to E_{g_{ba},\theta_b}$, which we denote by $\dagger$. If $p_{ab} = 0$, for $a = a_{n_1 n_2}(U_1)^{n_1}(U_2)^{n_2} \in E_{g_{ab},\theta_a} = \mathcal{A}_{\theta_a}$ $a^\dagger$ is simply

$$a^\dagger = (a_{n_1 n_2})^* (U_2)^{-n_2} (U_1)^{-n_1} \qquad \in \mathcal{A}_{\theta_a} \ ,$$

where $*$ is the complex conjugate. For $p_{ab} \neq 0$, the relation between $f_{ab} \in E_{g_{ab},\theta_a}$ and $f_{ab}^\dagger \in E_{g_{ba},\theta_b}$ is given by

$$f_{ab}(x, j) = (f_{ab}^\dagger)^* \left(\frac{x}{q_{ab} + p_{ab}\theta_a}, -j q_{ab} \right) \ . \tag{8}$$

[1] Here we change the notation of α_a and β_a from that in [28]. $q_a\alpha_a$ and $q_a\beta_a$ in this paper coincide with $-\alpha_a$ and β_a in [28], respectively.

Lemma 2.1 For $f_{ab} \in E_{g_{ab},\theta_a}$, $p_{ab} \neq 0$ and $f_{ab}^\dagger \in E_{g_{ba},\theta_b}$ defined in eq.(8),

$$(U_i(f_{ab}^\dagger))^* \left(\frac{x}{q_{ab}+p_{ab}\theta_a}, -jq_{ab} \right) = (f_{ab}Z_i)(x,j) \ ,$$
$$((f_{ab}^\dagger)Z_i)^* \left(\frac{x}{q_{ab}+p_{ab}\theta_a}, -jq_{ab} \right) = (U_i f_{ab})(x,j)$$

hold. Here note that U_i and Z_i in the left hand side act on E_{g_{ba},θ_b}, whereas, those in the right hand side act on E_{g_{ab},θ_a}.

It can be checked by a direct calculation. This lemma implies how the bijection $\dagger$ has good properties.

Definition 2.1 (Tr on $\mathcal{A}_{\theta_a}$) For $a = \sum_{(n_1,n_2)\in\mathbb{Z}^2} a_{n_1 n_2}(U_1)^{n_1}(U_2)^{n_2}$, $\mathrm{Tr}_{\mathcal{A}_{\theta_a}} : \mathcal{A}_{\theta_a} \to \mathbb{C}$ is defined by

$$\mathrm{Tr}_{\mathcal{A}_{\theta_a}} a := |q_a + p_a\theta| a_{00} \ . \tag{9}$$

Using this trace, the topological charge of noncommutative vector bundle $E_{g_a,\theta}$ over $\mathcal{A}_\theta$ is defined by $\mathrm{Tr}_{\mathcal{A}_{\theta_a}} \exp\left(\frac{[\nabla_1,\nabla_2]dx_1\wedge dx_2}{2\pi i}\right)$ where $dx_1 \wedge dx_2$ is the formal volume form on the noncommutative two-tori $\mathcal{A}_\theta$, and one gets $\mathrm{rank}(E_{g_a,\theta}) = |q_a + p_a\theta|$ and first Chern class (, *i.e.* , degree) $p_a \frac{q_a+p_a\theta}{|q_a+p_a\theta|}$.

Definition 2.2 Let E_a be a Heisenberg module $E_{g_a,\theta}$ equipped with a constant curvature connection (6) with $(\alpha,\beta) = (\alpha_a,\beta_a)$. We define the connection $\nabla_{ab,i}$ on $\mathrm{Hom}(E_a,E_b)$ by

$$\nabla_{ab,1} = \frac{1}{q_a+p_a\theta}\frac{\partial}{\partial x} - 2\pi i\beta_{ab} \ , \qquad \nabla_{ab,2} = -\frac{1}{q_a+p_a\theta}\frac{2\pi i p_{ab}x}{q_{ab}+p_{ab}\theta_a} - 2\pi i\alpha_{ab} \tag{10}$$

for $p_{ab} \neq 0$, and for $p_{ab} = 0$,

$$\begin{aligned} \nabla_{aa',1}(U_1)^{n_1}(U_2)^{n_2} &= 2\pi i \left(\frac{n_1}{q_a+p_a\theta} - \beta_{aa'} \right) (U_1)^{n_1}(U_2)^{n_2} \ , \\ \nabla_{aa',2}(U_1)^{n_1}(U_2)^{n_2} &= 2\pi i \left(\frac{n_2}{q_a+p_a\theta} - \alpha_{aa'} \right) (U_1)^{n_1}(U_2)^{n_2} \ , \end{aligned} \tag{11}$$

where

$$\alpha_{ab} := \frac{\alpha_b}{q_b+p_b\theta} - \frac{\alpha_a}{q_a+p_a\theta} \ , \qquad \beta_{ab} := \frac{\beta_b}{q_b+p_b\theta} - \frac{\beta_a}{q_a+p_a\theta} \ . \tag{12}$$

2.2 The tensor product of Heisenberg modules

Next, consider a tensor product between $\mathcal{A}_{g_a\theta}$-$\mathcal{A}_{g_b\theta}$ bimodule and $\mathcal{A}_{g_b\theta}$-$\mathcal{A}_{g_c\theta}$ bimodule. Namely, we construct a map φ

$$\varphi : \mathrm{Hom}(E_{g_a,\theta}, E_{g_b,\theta}) \otimes \mathrm{Hom}(E_{g_b,\theta}, E_{g_c,\theta}) \to \mathrm{Hom}(E_{g_a,\theta}, E_{g_c,\theta}) \ . \tag{13}$$

As stated in the previous subsection, $\mathrm{Hom}(E_{g_a,\theta}, E_{g_b,\theta})$ is equal to $\mathcal{A}_{\theta_a}$ if $p_{ab}=1$ and E_{g_{ab},θ_a} if $p_{ab}\neq 0$. The tensor product φ in eq.(13) is defined so that $\varphi(\mathrm{Hom}(E_{g_a,\theta}, E_{g_b,\theta})a\otimes\mathrm{Hom}(E_{g_b,\theta}, E_{g_c,\theta})) = \varphi(\mathrm{Hom}(E_{g_a,\theta}, E_{g_b,\theta})\otimes a\mathrm{Hom}(E_{g_b,\theta}, E_{g_c,\theta}))$ for $a\in\mathcal{A}_{\theta_b}$. If $p_{ab}=p_{bc}=0$, this tensor product is just the usual product in $\mathcal{A}_{\theta_a}$. If $p_{ab}=0$ and $p_{bc}\neq 0$, it is given by the action of $\mathcal{A}_{\theta_a}=\mathcal{A}_{\theta_b}$ on $E_{g_{bc},g_b\theta}$ (eq.(2)). In the case $p_{ab}\neq 0$ and $p_{bc}=0$, it is given by the right action of $\mathcal{A}_{\theta_b}=\mathcal{A}_{\theta_c}$ on $E_{g_{ab},g_a\theta}$ (eq.(3)). In the case $p_{ab}p_{bc}\neq 0$, if $g_{ab}g_{bc}=1$, the tensor product $\varphi: E_{g_{ab},\theta_a}\otimes E_{g_{ba},\theta_b}\to\mathcal{A}_{\theta_a}$ is given by

$$\varphi(f_{ab}\otimes f_{ba}) = \sum_{(n_1,n_2)\in\mathbb{Z}^2}(U_1)^{n_1}(U_2)^{n_2}\sum_{j\in\mathbb{Z}_{|p_{ab}|}}\int dx((U_2)^{-n_2}(U_1)^{-n_1}f_{ab}(x,j))f_{ba}(\frac{x}{q_{ab}+p_{ab}\theta_a},-q_{ab}j)\ . \tag{14}$$

Using eq.(8), one can also write this as

$$\varphi(f_{ab}\otimes f_{ba}) = \sum_{(n_1,n_2)\in\mathbb{Z}^2}(U_1)^{n_1}(U_2)^{n_2}\sum_{j\in\mathbb{Z}_{|p_{ab}|}}\int dx((U_2)^{-n_2}(U_1)^{-n_1}f_{ab}(x,j))((f_{ba}^\dagger)^*(x,j))\ .$$

This type of the tensor product is sometimes called an inner product in the context of Morita equivalence on noncommutative tori (see [51]).

Alternatively, for $g_{ab}g_{bc}\neq 1$, a tensor product was first constructed explicitly in [12] and modified in [28] for our purposes. [2] It is given by

$$\varphi(f_{ab}\otimes f_{bc})(x,j) = \sum_{u\in\mathbb{Z}}f_{ab}(x+\frac{1}{p_{ab}}(u-\frac{p_{bc}}{p_{ac}}j),-r_{ab}u+j)\cdot f_{bc}(\frac{x}{q_{ab}+p_{ab}\theta_a}-\frac{q_{bc}+p_{bc}\theta_b}{p_{bc}}(u-\frac{p_{bc}}{p_{ac}}j),u)\ . \tag{15}$$

In any case, the tensor product is defined so that it satisfies

$$\varphi((f_{ab}Z_i)\otimes f_{bc}) = \varphi(f_{ab}\otimes(U_if_{bc})) \tag{16}$$

for $i=1,2$, and further

$$\begin{aligned}\varphi((U_if_{ab})\otimes f_{bc}) &= U_i\varphi(f_{ab}\otimes f_{bc})\\ \varphi(f_{ab}\otimes(f_{bc}Z_i)) &= \varphi(f_{ab}\otimes f_{bc})Z_i\end{aligned} \tag{17}$$

for $i=1,2$. Here U_i's in the left hand side are those which act on E_{g_{ab},θ_a} and U_i's in the right hand side are those which act on E_{g_{ac},θ_c}. Similarly Z_i's in both sides are different from each other. By direct calculations one can check that the tensor product satisfies eq.(16) and (17).

Lemma 2.2 The tensor product φ is associative. Namely, for any $g_{ab},g_{bc},g_{cd}\in SL(2,\mathbb{Z})$ and $f_{ab}\in\mathrm{Hom}(E_a,E_b)$ etc. ,

$$\varphi(\varphi(f_{ab}\otimes f_{bc})\otimes f_{cd}) = \varphi(f_{ab}\otimes\varphi(f_{bc}\otimes f_{cd}))$$

holds.

[2] In [12] the tensor product between right $\mathcal{A}_\theta$ modules and left $\mathcal{A}_\theta$ modules is constructed. In that case, for right $\mathcal{A}_\theta$ module $E_{g_a,\theta}$ and left $\mathcal{A}_\theta$ modules $E_{g_b,\theta}$, the tensor product is defined so that $E_{g_a,\theta}U_i\otimes_{\mathcal{A}_\theta}E_{g_b,\theta}\sim E_{g_a,\theta}\otimes_{\mathcal{A}_\theta}U_iE_{g_b,\theta}$ where U_i's are two generators of $\mathcal{A}_\theta$. We instead define the tensor product as in eq.(16) for our purposes. See also [47], where the tensor product is of the same kind as ours, but in slightly different notations.

proof. The proof is essentially the same as those given by [47]. ■

Lemma 2.3 The connection on $\mathrm{Hom}(E_a, E_b)$ in Definition 2.2 satisfies the Leibniz rule with respect to the tensor product. Namely, for any a, b, c and $f_{ab} \in \mathrm{Hom}(E_a, E_b)$, $f_{bc} \in \mathrm{Hom}(E_b, E_c)$,

$$\nabla_{ac,i}\varphi(f_{ab} \otimes f_{bc}) = \varphi(\nabla_{ab,i}(f_{ab}) \otimes f_{bc}) + \varphi(f_{ab} \otimes \nabla_{bc,i}(f_{bc})) \tag{18}$$

holds.

Using eq.(14), we introduce two types of inner products between Heisenberg modules. One is given by the trace of the tensor product (14)

$$\langle f_{ab}, f_{ba} \rangle := \mathrm{Tr}\, \varphi(f_{ab} \otimes f_{ba}) \ , \tag{19}$$

where Tr is the one defined in Definition 2.1. Another one is, for $f_{ab}, f'_{ab} \in E_{g_{ab},\theta_a}$,

$$\langle f'_{ab} | f_{ab} \rangle := \mathrm{Tr}\, \varphi(f_{ab} \otimes (f'_{ab})^\dagger) \ . \tag{20}$$

Using the latter one, one can further define a norm. For $f_{ab} \in E_{g_{ab},\theta_a}$ it is defined by

$$||f_{ab}|| = (\langle f_{ab} | f_{ab} \rangle)^{\frac{1}{2}} \ . \tag{21}$$

Lemma 2.4 The inner product $\langle\ ,\ \rangle$ in (19) is symmetric, that is,

$$\langle f_{ba}, f_{ab} \rangle = \langle f_{ab}, f_{ba} \rangle \tag{22}$$

holds.

proof. This follows from eq.(14) and the definition of the trace (Definition 2.1).

2.3 T-duality for open strings and mirror symmetry of noncommutative two-tori

In this subsection, we discuss a physical aspect of our arguments. We propose a mirror symmetry for noncommutative two-tori, which is the background of our homological mirror symmetry set-up in the next section, and explain the relation of noncommutativity with the t-structure in D-brane stability.

Heisenberg modules discussed in subsection 2.1 and 2.2 can be thought of as noncommutative analogue of vector bundles over a noncommutative two-torus. Physically, a Heisenberg module $E_{g,\theta}$ with $\left(\begin{smallmatrix} q & s \\ p & r \end{smallmatrix}\right)$ is identified with a q D2-branes p D0-branes bound state on the noncommutative two-torus. Thus, by adding a notion of complex structure, one can consider a noncommutative analogue of a category of coherent sheaves on two-tori. Though these Heisenberg modules are based on the crossed product expression of a noncommutative two-torus $\mathcal{A}_\theta$, a noncommutative two-torus defined by deformation quantization (cf. [50]) may be better understood geometrically, since it reduces to a usual commutative two-torus in a commutative limit. In this picture, one can construct topologically twisted vector bundles over a noncommutative two-torus called quantum

twisted bundles ([23, 43, 4] and see [34]). Then, a Heisenberg module is in fact isomorphic to a quantum twisted bundle with compatibility of the structures of connections and so on (see [43, 34]). On the other hand, the crossed product representation of the noncommutative two-torus $\mathcal{A}_\theta$ and Heisenberg modules over it have a geometric realization which is directly related to the T-dual expression where open strings stretch between D1-branes [28]. Namely, one can say that the correspondence between the two descriptions are just T-duality. We shall give an explanation about it later in this subsection, and here we first recall this geometric realization of Heisenberg modules along [28].

The underlying geometry of the crossed product representation of the rotation algebras $\mathcal{A}_\theta$ is the Kronecker foliation[8]. Let $\hat{\mathbb{T}}^2$ be a two-torus with coordinates (x_1, x_2) with periodicity $x_1 \sim x_1 + 1$ and $x_2 \sim x_2 + 1$. Here we denote it by $\hat{\mathbb{T}}^2$ since it is the dual of a two-torus we consider later. For the covering space $\widetilde{\hat{\mathbb{T}}^2} \simeq \mathbb{R}^2$, by projections $\pi_1 : \widetilde{\hat{\mathbb{T}}^2} \to S^1 \otimes \mathbb{R}$ and $\pi_2 : \widetilde{\hat{\mathbb{T}}^2} \to \mathbb{R} \otimes S^1$ corresponding to the identifications $x_1 \sim x_1 + 1$ and $x_2 \sim x_2 + 1$, respectively, we have $\pi_{12}\widetilde{\hat{\mathbb{T}}^2} = \hat{\mathbb{T}}^2$ where $\pi_{12} := \pi_1\pi_2 = \pi_2\pi_1$. Let us consider the line

$$x_1 + \theta x_2 = 0 \tag{23}$$

on the covering space $\widetilde{\hat{\mathbb{T}}^2}$. When θ is irrational, the image of the line (23) by π_{12} fills densely in $\hat{\mathbb{T}}^2$. The pair of $\hat{\mathbb{T}}^2$ and the image of line (23) is called the Kronecker foliation with irrational slope θ, where the image of the line is called the leaf of the Kronecker foliation.

On the covering space $\widetilde{\hat{\mathbb{T}}^2}$, there are the mirror images of S^1 which are expressed as $x_2 = u_2 \in \mathbb{Z}$. $U_1 = e^{2\pi i x_1}$ is then the generator of functions on this S^1. The leaf is necessarily transversal to the S^1. Let us parametrize the leaf as $(x_1, x_2) = (\theta t + x, -t)$, $t \in \mathbb{R}$. The point $t = 0$ is an intersection point of the leaf and the S^1, and the leaf then intersects with the S^1 in the next time at $t = 1$, *i.e.* $(x_1, x_2) = (x + \theta, -1)$. Correspondingly, we can define a map $o_U : S^1 \to S^1$ by $o_U(x) = x + \theta$, and $U_2 : C(S^1) \to C(S^1)$ is regarded as the pullback;

$$(U_2 a)(x) = (o_U^*)a(x) = a(x + \theta) \ .$$

The cycle S^1 is identified with a D1-brane. An interval along the leaf, whose ends are on the D1-brane, is an open string. We shall justify this identification later. The state $(U_1)^{n_1}(U_2)^{n_2}$ is then the open string state which moves to x_1-direction by momentum n_1 and winds n_2 times around x_1-direction.

The situation above can be thought of the one for $E_{g,\theta}$, $g = \left(\begin{smallmatrix} q & s \\ p & r \end{smallmatrix}\right) \in SL(2, \mathbb{Z})$ with $p = 0$. Alternatively, a Heisenberg module $E_{g,\theta}$, $p \neq 0$ can be realized as follows. We first fix a line $L_{base} : x_2 = 0$, whose image by π_1 is the S^1 above. We regard it as the 'base space', and consider lines $L_g^j : q x_2 = p x_1 - q j$, $j = 0, \cdots, |p| - 1$ in $\widetilde{\hat{\mathbb{T}}^2}$ 'over' the base space. More precisely, we regard a function over a pair (L_{base}, L_g^j) as a 'vector bundle' over the S^1. For each j, the coordinate of L_g^j is identified with that of L_{base} by 'open strings'. Namely, for $x \in \mathbb{R}$ the coordinate of L_{base} such that $(x_1, x_2) = (qj/p + x, 0)$, we introduce the coordinate on L_g^j by $x \in \mathbb{R}$ such that

$$(x_1, x_2) = \left(\frac{qx}{q + p\theta} + \frac{qj}{p}, \frac{px}{q + p\theta} \right) \ . \tag{24}$$

Here we set the coordinate so that a point $x = 0$ is the intersection point of L_{base} and L_g^j. We call the intersection point the origin of (L_{base}, L_g^j). By definition, a point $(x_1, x_2) = (qj/p + x, 0)$ in L_{base} and a point $(x_1, x_2) = (qx/(q + p\theta) + qj/p, px/(q + p\theta)$ in L_g^j are the two ends of an interval along the leaf of the foliation. Let us consider a function f^j on it so that $f^j(x) = f(x, j)$ and $f \in \mathcal{S}(\mathbb{R} \times \mathbb{Z}_{|p|})$. Furthermore, we prepare infinite copies of pairs (L_{base}, L_g^j) by parallel-transforming them by $\mathbb{Z}^2$, where the origin is at $(x_1, x_2) = (qj/p, 0) + (u_1, u_2)$, $(u_1, u_2) \in \mathbb{Z}^2$ and the coordinate x is defined so that $x = 0$ describes the origin. To each of them, we associate a function $\tilde{f}^j(u_1, u_2)$ as $\tilde{f}(u_1, u_2; x, j) := \tilde{f}^j(u_1, u_2)(x) = f(x, j)$. We think of $\tilde{f}^j(u_1, u_2)$ a function on a line $(x_1, x_2) = (qj/p + u_1 + x, u_2)$ when U_i's act on it, whereas a function on a line $(x_1, x_2) = (qj/p + u_1 + qx/(q + p\theta), u_2 + px/(q + p\theta))$ when Z_i's act. For the operation of U_i's, corresponding to the periodicity $x_1 \sim x_1 + 1$ of the line $(x_1, x_2) = (qj/p + u_1 + x, u_2)$, the period of x is $x \sim x + 1$. The action of U_1 is essentially the multiplication of the generator of functions on the line with periodicity $x \sim x + 1$, that is, $U_1 \sim e^{2\pi i x}$. However, it is slightly modified so that $U_1(u_1, u_2; x + q/p, j) = U_1(u_1', u_2'; x, j + 1)$ holds since $(u_1, u_2; x + q/p, j)$ and $(u_1', u_2'; x, j + 1)$ denote the same point in $\hat{\mathbb{T}}^2$. In this way the action of U_1 is given by

$$U_1(u_1, u_2; x, j) = e^{2\pi i(x + j\frac{q}{p})} \; (= U_1(u_1', u_2'; x - \frac{q}{p}, j + 1) \;)$$

as multiplication on functions on $\mathbb{Z}^2 \times \mathbb{R} \times \mathbb{Z}_{|p|}$. Alternatively, the action of U_2 is defined as the pullback of $o_U : \mathbb{Z}^2 \times \mathbb{R} \times \mathbb{Z}_{|p|} \to \mathbb{Z}^2 \times \mathbb{R} \times \mathbb{Z}_{|p|}$, which we give by $o_U(u_1, u_2; x, j) = (u_1, u_2 - 1; x + q/p + \theta, j + 1)$ so that the origin is transformed from $(qj/p + u_1, u_2)$ to $(qj/p + u_1, u_2) - (q/p, 1) = (q(j-1)/p + u_1, u_2 - 1)$ and two points $(u_1, u_2; x, j)$, $((u_1, u_2 - 1; x + q/p + \theta, j - 1))$ are the two end points of an interval (open string) along the leaf. Thus, one obtains

$$(U_2\tilde{f})(u_1, u_2; x, j) = \tilde{f}(u_1, u_2 - 1; x + \frac{q}{p} + \theta, j - 1) \; .$$

On the other hand, for Z_i's, the period is $x \sim x + (q + p\theta)$ in the sense that x and $x + (q + p\theta)$ describe the same point in $\hat{\mathbb{T}}^2$. In addition, $(u_1, u_2; x + (q + p\theta)/p, j)$ and $(u_1', u_2'; x, j + 1)$ should represent the same point on the D1-brane. Thus, we define

$$Z_1(u_1, u_2; x, j) = e^{2\pi i(\frac{x}{q + p\theta} + \frac{j}{p})} \; .$$

The action of Z_2 is then the pullback of $o_Z : \mathbb{Z}^2 \times \mathbb{R} \times \mathbb{Z}_{|p|} \to \mathbb{Z}^2 \times \mathbb{R} \times \mathbb{Z}_{|p|}$, $o_Z(u_1, u_2; x, j) = (u_1 - s, u_2; x + 1/p, j - r)$ which transform the origin $(qj/p + u_1, u_2)$ to $(qj/p + u_1, u_2) - (1/p, 0) = (q(j-r)/p + u_1 - s, u_2)$ and an open string starting at $(u_1, u_2; x, j)$ ends at $(u_1 - s, u_2; x + 1/p, j - r)$. Then, one obtains

$$(\tilde{f}Z_2)(u_1, u_2; x, j) = \tilde{f}(u_1 - s, u_2; x + \frac{1}{p}, j - r) \; .$$

Then, projecting $\tilde{f}^j(u_1, u_2)$ to f^j by simply dropping indices (u_1, u_2), we get the Heisenberg module $E_{g,\theta}$ in subsection 2.1. One can see that the action of U_i and Z_i for the same $i = 1, 2$ commutes trivially, and the action of U_i and Z_i for different i commutes by the definition of the coordinate eq.(24). Note that these realizations of Heisenberg modules admit the ambiguity of parallel transformations on $\widetilde{\hat{\mathbb{T}}^2}$, which, we see in subsection 3.2, correspond to α_a.

Although we identify a Heisenberg module $E_{g,\theta}$ with a D1-brane winding cycle (q,p), it can be identified with the space of open strings binding between a $(0,1)$ D1-brane (base S^1) and a (p_a,q_a) D1-brane if it is treated as a $\mathcal{A}_\theta$-$\mathcal{A}_{\theta_a}$ bimodule. In more general, $\mathrm{Hom}(E_a,E_b)$ is regarded as the space of open strings stretching between D1-branes E_a and E_b (see [56, 30]). Since $\mathrm{Hom}(E_a,E_b) = E_{g_{ab},\theta_a}$ is characterized by two lines with slope p_a/q_a and p_b/q_b on the covering space, this situation can clearly be reduced to the case explained above by $SL(2,\mathbb{Z})$ translation $(g_a)^{-1}$ on $\hat{\mathbb{T}}^2$. The composition of Hom is then the interaction of open strings which is associative.

We called intervals binding between two D1-branes along the leaves of the foliation open strings. It is in fact justified as follows [28]. Now the noncommutativity of deformation quantization [3] is well understood from open string theory with B-field background (see [6, 53, 56]). Along this line, noncommutative two-tori is obtained by the algebra of correlation functions of open strings with both ends on D2-brane on a two-torus $\mathbb{T}^2$ [56]. The explanation is simplified if the Seiberg-Witten limit is taken [56], so we consider this situation. Let two by two matrix $E := g + B$ be the background of the $\mathbb{T}^2$ on which a D2-brane exists, where $g = \left(\begin{smallmatrix} g_{11} & g_{12} \\ g_{12} & g_{22}\end{smallmatrix}\right)$ is a positive definite metric and $B = \left(\begin{smallmatrix} 0 & -b \\ b & 0\end{smallmatrix}\right)$ is a shewsymmetric matrix called B-field. [3] It is known that in the Seiberg-Witten limit $g \to 0$ a noncommutative torus $\mathcal{A}_\theta$ with $\theta = -\frac{1}{b}$ is obtained .

Let us T-dualize for x_2-direction. The background E is transformed to $\hat{E}$ [20],

$$\hat{E} = (I_1 E + I_2)(I_2 E + I_1)^{-1} \ , \qquad I_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad I_2 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \ ,$$

or more explicitly, $\hat{E} := \hat{g} + \hat{B}$ is given by

$$\hat{g} + \hat{B} = g_{22}^{-1} \begin{pmatrix} \det(g) + b^2 & -b \\ -b & 1 \end{pmatrix} + g_{22}^{-1} \begin{pmatrix} 0 & g_{12} \\ -g_{12} & 0 \end{pmatrix} \ . \tag{25}$$

For open string theory, this T-duality transforms a D2-brane on $\mathbb{T}^2$ to a $(1,0)$D1-brane (D1-brane on the base S^1). In the Seiberg-Witten limit $g \to 0$, the above metric reduce to

$$\hat{g} \sim g_{22}^{-1} b^2 \begin{pmatrix} 1 & -\frac{1}{b} \\ -\frac{1}{b} & \frac{1}{b^2} \end{pmatrix} \ . \tag{26}$$

Open strings ending on the D1-brane take their configuration so that they have the minimum masses. Here, one can see that this metric is degenerate along the leaf with slope $-\frac{1}{\theta} = b$. By this T-duality transformation, a bound state of q D2-brane and p D0-brane, which we call a (q,p) D2-D0 brane, corresponds to a (q,p) D1-brane, a D1-brane on a geodesic cycle winding q-times around x_1-direction and p-times around x_2-direction. Open strings binding between any such two D1-branes stretch along the leaf and the situation coincides with the noncommutative two-tori associated with the Kronecker foliation. Consequently, we can conclude that the two representations of the noncommutative two-tori $\mathcal{A}_\theta$, that in the deformation quantization and that in the crossed product, are related by the T-duality [28].

[3] Do not confuse this metric g with $g \in SL(2,\mathbb{Z})$ used to define a Heisenberg module. There are no connection between them.

For two-tori, this T-duality is equivalent to the mirror symmetry [13]. The pair of complex structure τ and complexified symplectic structure ρ are defined in one-to-one correspondence with the backgrounds $E = g + B$ as

$$\tau = \frac{g_{12}}{g_{11}} + i\frac{\sqrt{g}}{g_{11}} , \qquad \rho = i\sqrt{g} + b . \tag{27}$$

The mirror dual torus is defined so that the complex structure and the complexified symplectic structure are interchanged. However, we have an ambiguity of the identification which comes from an automorphism $SL(2,\mathbb{Z})$ acting compatibly on each two-torus (see [26]). One can check that, by our choice of T-duality (25), the complex structure and the complexified symplectic structure are transformed as

$$\hat{\tau} = -\frac{1}{\rho} , \qquad \hat{\rho} = -\frac{1}{\tau} . \tag{28}$$

In this paper we define $\hat{\tau}$ and $\hat{\rho}$ the complex structure and the complexified symplectic structure, respectively, on the mirror dual torus. The homological mirror of (commutative) two-tori then claims an equivalence of a category $\mathcal{C}^B$ of coherent sheaves on $\mathbb{T}^2$ (B indicates the corresponding topological string theory is the B-model [64]) and the Fukaya's A_∞ category $\mathcal{C}^A$ on the mirror dual two-torus $\hat{\mathbb{T}}^2$ (A-model side) for a fixed $\hat{\rho} = -\frac{1}{\tau}$ (and vice visa). The category of coherent sheaves depends only on the complex structure τ and is independent of the complexified symplectic structure ρ, while the A_∞-category depends only on the complexified symplectic structure $\hat{\rho}$. Especially, for our two-tori case, the coherent sheaves, the objects of $\mathcal{C}^B$, are identified with D2-D0 brane bound states, whereas the objects of $\mathcal{C}^A$ are the D1-branes on the geodesic cycles. As stated previously, from a open string physics viewpoint, both $\mathbb{T}^2$ and $\hat{\mathbb{T}}^2$ are associated with noncommutative two-tori $\mathbb{T}^2_\theta$ of the same θ but different descriptions, especially in the Seiberg-Witten limit $g \to 0$. Though the Heisenberg modules are interpreted from the geometry of crossed product representation of noncommutative two-torus $\mathbb{T}^2_\theta = \mathcal{A}_\theta$, we treat them as the T-dual side, the B-model on $\mathbb{T}^2_\theta$. Namely, by defining a complex and holomorphic structures on $\mathbb{T}^2_\theta$ we can consider a noncommutative version of $\mathcal{C}^B$, which we denote by $\mathcal{C}^B_\theta$. On the other hand, by the T-dual correspondence we can expect what should be taken for the mirror side, $\mathcal{C}^A_\theta$. As a result, it is essentially equivalent to $\mathcal{C}^A$, the commutative case, and some correspondence of $\mathcal{C}^B_\theta$ and $\mathcal{C}^A_\theta$ is in fact observed explicitly in [28]. Here, the Seiberg-Witten limit corresponds to the boundary of the upper half plane, the moduli of ρ. The shape of the dual torus is degenerate since $\hat{\tau} = \theta$. In order to extend the correspondence between closed string variables (τ, ρ) (or $g + B$) and the noncommutativity θ to the general situation $g \neq 0$, we need one more parameter ϕ so that the action of Morita equivalence acts compatibly. The correspondence is then given as follows [44];

$$(g + B)^{-1} = (G + \Phi)^{-1} + \Theta , \tag{29}$$

where $\Phi := \begin{pmatrix} 0 & -\phi \\ \phi & 0 \end{pmatrix}$, $\phi \in \mathbb{R}$ and $\Theta := \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}$. If we set $g \to 0$ and $\phi = 0$, we recover the previous situation (26). One can also read the identity (29) as the relation between the moduli $g + B$ and $G + \Phi$ for a fixed θ. In fact there exists a bijection between the moduli $g + B$ and $G + \Phi$. One may identify G and Φ with the metric and the B-field on the noncommutative two-torus $\mathbb{T}^2_\theta$,

respectively. Thus, one can define a complex structure and a complexified symplectic structure on $\mathbb{T}^2_\theta$ in a similar way;

$$\tau_\theta = \frac{G_{12}}{G_{11}} + i\frac{\sqrt{G}}{G_{11}} \ , \qquad \rho_\theta = i\sqrt{G} + \phi \ , \tag{30}$$

and the mirror symmetry for noncommutative two-tori can be defined by ;

$$\hat{\tau}_\theta = -\frac{1}{\rho_\theta} \ , \qquad \hat{\rho}_\theta = -\frac{1}{\tau_\theta} \ . \tag{31}$$

This is in fact a $\mathbb{Z}_2$ symmetry. Thus, the mirror symmetry is extended on the one parameter family θ. Then, what is the mirror dual noncommutative two-torus ? It is identified with the (commutative) two-torus with complex structure $\hat{\tau}_\theta$ and complexified symplectic structure $\hat{\rho}_\theta$ but foliated by lines of slope characterized by θ. We denote this two-torus by $\hat{\mathbb{T}}^2(\hat{\tau}_\theta, \hat{\rho}_\theta; \theta)$. Let us express them on the complex plane with coordinate $x + iy$, where the two-torus is described by periods $(x, y) = (1, 0)$ and $(x, y) = (\mathrm{Re}(\hat{\tau}_\theta), \mathrm{Im}(\hat{\tau}_\theta))$. Then the leaf of the foliation is defined by parallel transformations of line L_θ,

$$L_\theta : \mathrm{Im}(\hat{\tau}_\theta)x + \theta y = 0 \ . \tag{32}$$

On the other hand, by a direct calculation one can see that τ is invariant under the change of θ, $\tau_\theta = \tau$. Namely, for two-tori, the homological mirror symmetry on this noncommutative set-up should be essentially the same as the commutative case. The only θ-dependence comes through t-structure of the D-brane stability [14, 5]. The basic tool in the D-brane stability is the central charge $\mathcal{Z}$. For commutative two-tori, it can be defined by

$$\mathcal{Z}_\rho(E_{g_a,\theta=0}) = \frac{1}{g_s}\left(\rho q_a - p_a\right) \ , \tag{33}$$

where $g_s \in \mathbb{C}$ is the complexified string coupling constant. We call $E_{g_a,\theta}$ the D-brane a, and sometimes write the corresponding central charge as $\mathcal{Z}_\rho(E_{g_a,\theta=0}) = \mathcal{Z}_\rho(a)$. The absolute value defines the lowest bound of D-brane bound states with rank $= q_a$ and deg $= p_a$.

$$|\mathcal{Z}_\rho(a)| = \frac{|q_a|}{|g_s|}\sqrt{\det\left(g + B - \frac{p_a}{q_a}\right)} \ . \tag{34}$$

This is nothing but the mass of the D-brane with constant curvature connections. Alternatively, in the variables of the dual torus $\hat{\mathbb{T}}^2(\hat{\tau}_\theta, \hat{\rho}_\theta; \theta)$, one can see that the central charge (33) is expressed as

$$\mathcal{Z}_{\hat{\tau}}(a) \propto (q_a + \hat{\tau} p_a) \ . \tag{35}$$

In an appropriate definition of the complexified string coupling, the absolute value again gives the mass of (q, p) D1-brane on $\hat{\mathbb{T}}^2(\hat{\tau}_\theta, \hat{\rho}_\theta; \theta)$. The angle of $q_a + \hat{\tau} p_a$ is just the angle of D1-brane a. If we consider the sum $\mathcal{Z}_{\hat{\tau}}(a) + \mathcal{Z}_{\hat{\tau}}(b)$ corresponding to the direct sum of the Heisenberg modules a and b, in the dual D1-brane picture it is described by two segments passing through 0, $q_a + \hat{\tau} p_a$ and $(q_a + q_b) + \hat{\tau}(p_a + p_b)$. Of course the D1-brane of rank $= q_a + q_b$, deg $= p_a + p_b$

with minimum mass is described by the straight segment between 0 and $(q_a+q_b)+\hat{\tau}(p_a+p_b)$. Thus, for $n_c(q_c,p_c)=(q_a+q_b,p_a+p_b)$ with $n_c\in\mathbb{Z}$ and relatively prime integers (q_c,p_c), we get

$$|\mathcal{Z}_\rho(a)|+|\mathcal{Z}_\rho(b)|\geq|\mathcal{Z}_\rho(a)+\mathcal{Z}_\rho(b)|=n_c|\mathcal{Z}_\rho(c)|\ . \tag{36}$$

This is just the stability bound.

We can extend these central charges on one parameter θ by the relation (29) with preserving all theses structures. The central charge is defined on $\mathbb{T}^2_\theta$ by

$$\mathcal{Z}_{\rho_\theta}(E_{g_a,\theta})=\frac{1}{G_s}\left(\rho_\theta q_a-p_a\right)\ , \tag{37}$$

where $G_s\in\mathbb{C}$ is the complexified (effective) string coupling constant. In this definition, the absolute value takes the following form,

$$|\mathcal{Z}_{\rho_\theta}(E_{g_a,\theta})|=\mathrm{Tr}_{\mathcal{A}_\theta}\frac{1}{|G_s|}\sqrt{\det\left(G+\Phi-\frac{p_a}{q_a}\right)} \tag{38}$$

which is in fact independent of θ and so especially equal to $\mathcal{Z}_\rho(E_{g_a,\theta=0})$. Namely, one can define the effective string coupling constant G_s so that the D-brane masses are preserved for all p and q. This is just the D-brane mass discussed in the context of noncommutative solitons in [30] (for solitons on noncommutative two-tori see also [37]). Then, the stability bound condition (36) remains to hold. The equality holds only if $p_a/q_a=p_b/q_b$ for general ρ. In this case, E_c is called *semi-stable*. However notice that, in the degenerate limit $g\to 0$, *i.e.* , $\mathrm{Im}(\rho)\to 0$, we have $|\mathcal{Z}_\rho(a)|+|\mathcal{Z}_\rho(b)|=|\mathcal{Z}_\rho(a)+\mathcal{Z}_\rho(b)|$ for all a and b. This means, any D-brane can be decomposed into smaller D-branes. This phenomena is just the instability argued in [2]. Note that $E_{g,\theta}$ and $E_{-g,\theta}$, in the dual side $\hat{\mathbb{T}}^2$, describe the D1-branes with the same angles. However, they have the D-brane charges of opposite signs, (q,p) and $(-q,-p)$. They are relatively distinguished, one is a D-brane, and another one is an anti-D-brane. Namely, for a fixed $E_{g,\theta}$, an open string starting from it distinguishes whether the endpoint belong to $E_{g',\theta}$ or $E_{-g',\theta}$. Let us define the slope of D-brane a by

$$-\pi<\mathrm{Arg}(a)\leq\pi\ ,\qquad \mathrm{Arg}(a):=\mathrm{Im}\log\left(\frac{\mathcal{Z}_{\hat{\tau}_\theta}(a)}{\mathcal{Z}_{\hat{\tau}_\theta}(E_{g=1,\theta})}\right)=\mathrm{Im}\log(q_a+\hat{\tau}_\theta p_a)\ . \tag{39}$$

In particular $\mathrm{Arg}(a)=0$ for $g_a=1$. Since whether a is a D-brane or an anti-D-brane is distinguished only relatively now, we need to label the distinction by hand. This is the t-structure (for more general set-up see [5]). It should be good to identify the t-structure with noncommutativity θ ; let us define the set of D-branes (not anti-D-branes) by

$$\mathrm{Ob}(\mathcal{C}_\theta):=\{a\ |\mathrm{Arg}(-L_\theta)<\mathrm{Arg}(a)\leq\mathrm{Arg}(L_\theta)\}\ , \tag{40}$$

where, for the line L_θ in eq.(32) we set $\mathrm{Arg}(-L_\theta)$ and $\mathrm{Arg}(L_\theta)$ by

$$-\pi<\mathrm{Arg}(-L_\theta):=\mathrm{Im}\log(-i\mathrm{Im}(\hat{\tau}_\theta)+\theta)<0\ ,\qquad 0<\mathrm{Arg}(L_\theta):=\mathrm{Im}\log(i\mathrm{Im}(\hat{\tau}_\theta)-\theta)<\pi\ .$$

In particular, for $\hat{\mathbb{T}}^2_\theta$ of $\phi=\theta=0$, we have $\mathrm{Arg}(-L_\theta)=-\pi/2$ and $\mathrm{Arg}(L_\theta)=\pi/2$. The important point is that, for a fixed background $g+B$, the set $\mathrm{Ob}(\mathcal{C}^A_\theta)$ $(=\mathrm{Ob}(\mathcal{C}^B_\theta))$ is independent

of θ. Namely, $\hat{\mathbb{T}}^2(\hat{\tau}_\theta,\hat{\rho}_\theta;\theta)$ with different θ can be related by a $GL(2,\mathbb{R})_+$ transformation. This is the consequence obtained by translating the argument in [24] into our set-up. In this way, a noncommutative deformation is understood as a one-parameter extension of the mirror symmetry set-up preserving the structure of the corresponding 'D-brane categories' if we fix the background $g+B$, whereas it can be regarded as a deformation of t-structures if we fix the noncommutative background $G+\Phi$. In the next section, we concentrate on the case $\phi=0$ for simplicity, and propose the corresponding homological mirror set-up as a correspondence between a θ-deformed Fukaya category and a category of Heisenberg modules equipped with holomorphic structures. Since τ_θ is independent of θ as stated previously, hereafter we drop it and write simply as $\tau_\theta=\tau$, hence $\hat{\rho}_\theta=\hat{\rho}$.

3 On homological mirror on noncommutative two-tori

In this section we introduce a complex structure on noncommutative two-tori and discuss a noncommutative extension of homological mirror symmetry on (commutative) two-tori [35, 48, 45, 46] which has its background in the physics in the previous subsection.

In subsection 3.1 we recall the holomorphic structure on Heisenberg modules introduced by A. Schwarz [55, 12] and construct a cyclic differential graded (DG) category of (stable) holomorphic vector bundles over a noncommutative two-torus partially based on the work [47]. In subsection 3.2 we propose a candidate of the category on A-model side, the Fukaya's A_∞-category for noncommutative two-tori, and set up a homological mirror symmetry conjecture on noncommutative two-tori. As a part of the conjecture, we prove a categorical mirror symmetry. Finally in subsection 3.3 we complete the Feynman rule to generate higher Massey products, which form a cyclic A_∞-category, from the cyclic DG category of holomorphic vector bundles on noncommutative two-tori.

3.1 Holomorphic structure on noncommutative two-tori

We have seen that the Heisenberg modules are equipped the constant curvature connection as in eq.(6). Now, on noncommutative two-tori let us introduce a complex structure $\tau\in\mathbb{C}$ and and consider the following combination

$$\bar{\nabla}_a := \nabla_{a,1}-\frac{1}{\tau}\nabla_{a,2} \tag{41}$$

according to [55, 12]. $\bar{\nabla}_a$ is regarded as the noncommutative analogue of the holomorphic structure. The solutions of the following equation

$$\bar{\nabla}_a f(x,j)=0$$

exists iff $\frac{p_a}{q_a+p_a\theta}>0$, and they are given by

$$f(x,j)=a^j\exp\left(\frac{2\pi i p_a}{q_a+p_a\theta}\Big[\frac{\hat{\rho}}{2}\left(x+\frac{\alpha_a}{p_a}\right)^2+\frac{\beta_a}{p_a}\left(x+\frac{\alpha_a}{p_a}\right)\Big]\right)\ ,\qquad a^j\in\mathbb{C}\ , \tag{42}$$

where we denoted $-\frac{1}{\tau} =: \hat{\rho}$. They are called theta vectors or holomorphic vectors, since they span a basis of a $|p|$-dimensional vector space in $E_{g_a,\theta}$ [12]. One can see that $E_{g_a,\theta}$ and $E_{-g_a,\theta}$ is equivalent, in the sense that the holomorphic vector (42) is invariant under the replacement of (q_a, p_a, α_a) with $(-q_a, -p_a, -\alpha_a)$. Hereafter, in a reason discussed in the previous subsection we take Heisenberg modules $E_{g_a,\theta}$ with $q_a + p_a\theta > 0$.

We can define a holomorphic structure $\bar{\nabla}_{ab} : \mathrm{Hom}(E_a, E_b) \to \mathrm{Hom}(E_a, E_b)$ in a similar way;

$$\bar{\nabla}_{ab} := \nabla_{ab,1} + \hat{\rho}\nabla_{ab,2} \ . \tag{43}$$

Based on this structure, we construct a cyclic differential graded (DG) category of holomorphic vector bundles below. A cyclic DG category is a natural extension of a cyclic differential graded algebra (DGA) and defined as follows.

Definition 3.1 (Cyclic differential graded category) A differential graded (DG) category $\mathcal{C}$ consists of a set of objects $\mathrm{Ob}(\mathcal{C}) = \{a, b, \cdots\}$, a $\mathbb{Z}$-graded vector space $V_{ab} = \oplus_k V_{ab}^k$ for each two objects a, b and grading $k \in \mathbb{Z}$, a degree one differential $d : V_{ab}^k \to V_{ab}^{k+1}$ and a composition of morphisms $\varphi : V_{ab}^k \otimes V_{bc}^l \to V_{ac}^{k+l}$ satisfying the following relations ;

$$(d)^2 = 0 \ , \tag{44}$$

$$d\varphi(v_{ab} \otimes v_{bc}) = \varphi(d(v_{ab}) \otimes v_{bc}) + (-1)^{|v_{ab}|}\varphi(v_{ab} \otimes d(v_{bc})) \ , \tag{45}$$

$$\varphi(\varphi(v_{ab} \otimes v_{bc}) \otimes v_{cd}) = \varphi(v_{ab} \otimes \varphi(v_{bc} \otimes v_{cd})) \ , \tag{46}$$

where $|v_{ab}|$ is the degree of v_{ab}, that is, $v_{ab} \in V_{ab}^{|v_{ab}|}$. Let η be a nondegenerate symmetric inner product of fixed degree $|\eta| \in \mathbb{Z}$ on $V := \oplus_{a,b} V_{ab}$. Namely, for each a and b,

$$\eta : V_{ab}^k \otimes V_{ba}^l \to \mathbb{C} \tag{47}$$

is nondegenerate, nonzero only if $k + l + |\eta| = 0$, and satisfies $\eta(V_{ab}^k, V_{ba}^l) = (-1)^{kl}\eta(V_{ba}^l, V_{ab}^k)$. In this situation, we call a DG category with inner product η a *cyclic DG category* $\mathcal{C}$ if the following conditions hold;

$$\eta(dv_{ab}, v_{ba}) + (-1)^{|v_{ab}|}\eta(v_{ab}, dv_{ba}) = 0 \ , \tag{48}$$

$$\eta(\varphi(v_{ab} \otimes v_{bc}), v_{ca}) = (-1)^{|v_{ab}|(|v_{bc}|+|v_{ca}|)}\eta(\varphi(v_{bc} \otimes v_{ca}), v_{ab}) \ . \tag{49}$$

Also, we call a cyclic DG category $\mathcal{C}$ with $d = 0$ a *cyclic graded category.*

Remark 3.1 A (cyclic) DG category $\mathcal{C}$ consisting of one object only is called a (cyclic) differential graded algebra (DGA). On the other hand, if the space of morphisms $V = \oplus_{a,b} V_{ab}$ is thought of as a $\mathbb{Z}$-graded vector space, (V, d, φ) and (V, d, φ, η) can be regarded as a DGA and a cyclic DGA, respectively. Hereafter we sometimes express explicitly as (V, d, φ) or (V, d, φ, η) and call it a (cyclic) DG category. Similarly, we denote a cyclic graded category by triple (V, φ, η).

Proposition 3.1 (Cyclic DG category of holomorphic vector bundles) Consider a set $\{a, b, \cdots\}$ where each element $a = (g_a, \alpha_a, \beta_a)$ is a Heisenberg module with a constant curvature

connection E_a (see Definition 2.2) satisfying $q_a + p_a\theta > 0$. For each pair (a, b), we define a graded vector space of degree zero by $V_{ab}^0 = \mathrm{Hom}(E_{g_a,\theta}, E_{g_b,\theta})$, a graded vector space of degree one by $V_{ab}^1 = \mathrm{Hom}(E_{g_a,\theta}, E_{g_b,\theta}) \otimes d\bar{z}$, where $d\bar{z}$ is a formal degree one base of the antiholomorphic 1-forms on the noncommutative two-tori $\mathcal{A}_\theta$, and $V_{ab}^k = 0$ for $k \neq 0, 1$. On the graded vector space $V = \oplus_{a,b} \oplus_k V_{ab}^k$, a differential is defined by

$$dv_{ab} := (\bar{\nabla}_{ab} v_{ab}) \otimes d\bar{z} \ . \tag{50}$$

Also, extending the tensor product φ to $\otimes d\bar{z}$ trivially, one gets a tensor product $\varphi : V_{ab} \otimes V_{bc} \to V_{ac}$. An inner product $\eta : V_{ab}^k \otimes V_{ba}^l \to \mathbb{C}$ is given by

$$\eta = \mathrm{Tr} \int dz \varphi \tag{51}$$

where we set $\int dz d\bar{z} = 1$, so $\eta(V_{ab}^k, V_{ba}^l)$ is nonzero only if $k + l = 1$. Then (V, d, φ, η) forms a cyclic DG category.

proof. Since $(d\bar{z})^2 = 0$, it is clear that d is a differential (44). The Leibniz rule (45) and the associativity (46) just follow from the Leibniz rule for $\nabla_{ab,i}$ in Lemma 2.3 and associativity in Lemma 2.2, respectively, in subsection 2.2. By the definition of the inner product (51), it is in fact nondegenerate and symmetric due to eq.(22). The condition (48) then follows from Leibniz rule (Lemma 2.3), and the cyclicity (49) follow from Lemma 2.3 and eq.(22). ∎

Definition 3.2 For each objects a we define a number

$$\mu_a = \frac{p_a}{q_a + p_a\theta} \in \mathbb{R} \ . \tag{52}$$

Furthermore, for each a and b we set

$$\mu_{ab} := \frac{p_{ab}}{q_{ab} + p_{ab}\theta_a} \tag{53}$$

which characterizes properties of $\mathrm{Hom}(a, b)$.

Remark 3.2 We have the following identity

$$\mu_b - \mu_a = (q_a + p_a\theta)^{-2} \mu_{ab} \ , \tag{54}$$

so the sign of $\mu_b - \mu_a$ coincides with the sign of μ_{ab}.

The dimension of the cohomologies H^0 and H^1 are given as follows [47].

- For $\mu_{ab} > 0$, $\dim H^0(V_{ab}) = |p_{ab}|$ and $\dim H^1(V_{ab}) = 0$.
- For $\mu_{ab} < 0$, $\dim H^0(V_{ab}) = 0$ and $\dim H^1(V_{ab}) = |p_{ab}|$.
- For $\mu_{ab} = 0$, $\dim H^0(V_{ab}) = \dim H^1(V_{ab}) = 0$ for $a \neq b$ and $\dim H^0(V_{ab}) = \dim H^1(V_{ab}) = 1$ for $a = b$.

The cohomology $H^0(V_{ab})$ and $H^1(V_{ab})$ are the kernel and the cokernel of d, respectively. For $\mu_{ab}>0$, since $H^0(V_{ab})$ is the kernel of

$$\bar{\nabla}_{ab}=\frac{1}{q_a+p_a\theta}\left(\frac{\partial}{\partial x}-\frac{2\pi i\hat{\rho}p_{ab}x}{q_{ab}+p_{ab}\theta_a}\right)-2\pi i(\beta_{ab}+\hat{\rho}\alpha_{ab})\ ,$$

its base $|0\rangle_{ab}^j$, $j=0,\cdots,|p_{ab}|-1$ is given by

$$|0\rangle_{ab}^j(x):=\exp\left(\frac{2\pi i p_{ab}}{q_{ab}+p_{ab}\theta_a}\Big[\frac{\hat{\rho}}{2}\left(x+\frac{q_b+p_b\theta}{p_{ab}}\alpha_{ab}\right)^2+\frac{q_b+p_b\theta}{p_{ab}}\beta_{ab}\left(x+\frac{q_b+p_b\theta}{p_{ab}}\alpha_{ab}\right)\Big]\right)\ , \tag{55}$$

whose norm (defined by eq.(21)) is $(\frac{q_{ab}+p_{ab}\theta_a}{Tp_{ab}})^{\frac{1}{4}}$, where $T=-i(\hat{\rho}-\hat{\rho}^*)=2\mathrm{Im}(\hat{\rho})$.

In case $\mu_{ab}<0$, $H^0=0$ but H^1 is a $|p|$ dimensional vector space. Using the inner product, we can now take the cohomology element explicitly.

For $\bar{\nabla}_{ab}$ in eq.(43), define

$$\bar{\nabla}_{ab}^{\dagger}=\frac{1}{q_a+p_a\theta}\left(-\frac{\partial}{\partial x}+\frac{2\pi i\hat{\rho}^*p_{ab}x}{q_{ab}+p_{ab}\theta_a}\right)+2\pi i(\beta_{ab}+\hat{\rho}^*\alpha_{ab})\ . \tag{56}$$

By the definition of the inner product, we get ;

Lemma 3.1 For $f,f'\in E_{g_{ab},\theta_a}$,

$$\langle f'|\bar{\nabla}f\rangle=\langle\bar{\nabla}^{\dagger}f'|f\rangle\ ,\qquad \langle f'|\bar{\nabla}^{\dagger}f\rangle=\langle\bar{\nabla}f'|f\rangle \tag{57}$$

hold.

Thus, the cokernel of V_{ab}^1 is defined by the states annihilated by $\bar{\nabla}^{\dagger}$. It is spanned by $|0\rangle_{ab}^j\otimes d\bar{z}$ with

$$|0\rangle_{ab}^j(x):=\exp\left(\frac{2\pi i p_{ab}}{q_{ab}+p_{ab}\theta_a}\Big[\frac{\hat{\rho}^*}{2}\left(x+\frac{q_b+p_b\theta}{p_{ab}}\alpha_{ab}\right)^2+\frac{q_b+p_b\theta}{p_{ab}}\beta_{ab}\left(x+\frac{q_b+p_b\theta}{p_{ab}}\alpha_{ab}\right)\Big]\right)\ . \tag{58}$$

Now we have

$$[\bar{\nabla}_{ab},\bar{\nabla}_{ab}^{\dagger}]=\frac{2\pi Tp_{ab}}{(q_a+p_a\theta)(q_b+p_b\theta)}\ . \tag{59}$$

By employing this commutation relation, the space V_{ab} is expanded by the Hermite polynomials as follows (cf.[42, 47]). For V_{ab}^k with $\mu_{ab}>0$, define the $|p_{ab}|$-degenerate vacuum states $|0,k\rangle_{ab}^j$ by those annihilated by $\bar{\nabla}_{ab}$. We set

$$|0,0\rangle_{ab}^j=|0\rangle_{ab}^j\ ,\qquad |0,1\rangle_{ab}^j=|0\rangle_{ab}^j\otimes d\bar{z}\ . \tag{60}$$

Then, V_{ab}^k is spanned by

$$|n,k\rangle_{ab}^j:=(\bar{\nabla}^{\dagger})^n|0,k\rangle_{ab}^j\ ,$$

and $|n,k\rangle_{ab}^j$ is expressed by the Hermite polynomials ;

$$|n,k\rangle_{ab}^j(x) = \left(\frac{\pi T p_{ab}}{(q_a+p_a\theta)(q_b+p_b\theta)}\right)^{\frac{n}{2}} H_n\left(\sqrt{\frac{\pi T p_{ab}}{q_{ab}+p_{ab}\theta_a}}\left(x+\frac{q_b+p_b\theta}{p_{ab}}\alpha_{ab}\right)\right)|0,k\rangle_{ab}^j(x)\ ,$$
$$H_n(x) := \left(-\frac{\partial}{\partial x}+2x\right)^n\cdot 1\ . \tag{61}$$

Alternatively, V_{ab}^k with $\mu_{ab}<0$ are spanned by the states $|n,k\rangle_{ab}^j := (\bar\nabla)^n|0,k\rangle_{ab}^j$ where the vacuum states are defined by $|0,k\rangle_{ab}^j = |0\rangle_{ab}^j\otimes d\bar z^{\otimes k}$ with $|0\rangle_{ab}^j$ in eq.(58). Just as the Hermite polynomial expansion in eq.(61), each state is written as

$$|n,k\rangle_{ab}^j(x) = \left(\frac{-\pi T p_{ab}}{(q_a+p_a\theta)(q_b+p_b\theta)}\right)^{\frac{n}{2}} H_n\left(\sqrt{\frac{-\pi T p_{ab}}{q_{ab}+p_{ab}\theta_a}}\left(x+\frac{q_b+p_b\theta}{p_{ab}}\alpha_{ab}\right)\right)|0,k\rangle_{ab}^j(x)\ . \tag{62}$$

Thus, each V_{ab}^k is spanned by the linear combination of these states, and especially the cohomology states are spanned by eq.(55) and (58).

Remark 3.3 By comparing $|0\rangle_{ab}^j$ in eq.(55)and (58), and also eq.(61) and (62), one can see that $|n,k\rangle_{ab}^j$ with $\mu_{ab}>0$ and $|n,k\rangle_{ab}^j$ with $\mu_{ab}<0$ is related by the interchange of $\hat\rho$ and $\hat\rho^*$. This interchange leads the interchange of T and $-T$.

Finally, the case of a,a' with $p_{aa'}=0$ is easy. In this case we have $V_{aa'}^k\simeq\mathcal{A}_{\theta_a}\otimes(d\bar z)^{\otimes k}$, which is spanned by

$$|n,0\rangle_{aa'} = (U_1)^{n_1}(U_2)^{n_2}\ ,\qquad |n,1\rangle_{aa'} = (U_2)^{-n_2}(U_1)^{-n_1}\otimes d\bar z\ , \tag{63}$$

where $n=(n_1,n_2)\in\mathbb{Z}^2$. By definition we get $(|n,0\rangle_{aa'})^\dagger\otimes d\bar z = |n,1\rangle_{aa'}$. Since we have

$$\bar\nabla_{aa'}(U_1)^{n_1}(U_2)^{n_2} = 2\pi i\left(\left(\frac{n_1}{q_a+p_a\theta}-\beta_{aa'}\right)+\hat\rho\left(\frac{n_2}{q_a+p_a\theta}-\alpha_{aa'}\right)\right)(U_1)^{n_1}(U_2)^{n_2}\ ,$$

it is clear that only when $a=a'$ the cohomology is nonzero and $\dim H^k(V_{aa})=1$ for $k=0,1$. We denote these basis by $1 =: |0,0\rangle_{aa}$ and $1\otimes d\bar z =: |0,1\rangle_{aa}$.

Remark 3.4 For any a and b, the compatibility of the inner product with $\bar\nabla$, $\bar\nabla^\dagger$ leads the following relation

$$\left(|n,k\rangle_{ab}^j\right)^\dagger = |n,k\rangle_{ba}^{-q_{ab}j}\ . \tag{64}$$

Comparing this with eq.(20), we have the following.

Corollary 3.1 Any element $f_{ab}\in V_{ab}^k$ is expanded as

$$f_{ab} = \sum_n |n,k\rangle_{ab}^j N_{ab}(n)\eta(f_{ab},|n,1-k\rangle_{ba}^{-q_{ab}j})\ , \tag{65}$$

where $N_{ab}(n)$ is the normalization and given by

$$\begin{array}{rl} \sqrt{\frac{Tp_{ab}}{(q_a+p_a\theta)(q_b+p_b\theta)}}\frac{1}{[\bar{\nabla}_{ab},\bar{\nabla}^\dagger_{ab}]^n n!} & \text{for} \quad \mu_{ab}>0\ , \\ \sqrt{\frac{Tp_{ab}}{(q_a+p_a\theta)(q_b+p_b\theta)}}\frac{1}{[\bar{\nabla}_{ab},\bar{\nabla}^\dagger_{ab}]^n n!} & \text{for} \quad \mu_{ba}<0\ , \\ \frac{1}{q_a+p_a\theta} & \text{for} \quad \mu_{aa'}=0\ . \end{array} \tag{66}$$

Definition 3.3 (Hodge-Kodaira decomposition) Now, for any a,b, V^k_{ab} is spanned by $|n,k\rangle$, where $n\in\mathbb{Z}_{\geq 0}$ for $\mu_{ab}\neq 0$ and $n=(n_1,n_2)\in\mathbb{Z}^2$ for $\mu_{ab}=0$. In any case, we have $|n,k\rangle=0$ if $k\neq 0,1$. We define d^{-1} by

$$d^{-1}|n,k\rangle^j_{ab}=\frac{1}{(n+1)[\bar{\nabla}_{ab},\bar{\nabla}^\dagger_{ab}]}|n+1,k-1\rangle^j_{ab}$$

for $\mu_{ab}>0$,

$$d^{-1}|n,k\rangle^j_{ba}=|n-1,k-1\rangle^j_{ba}\ (n\geq 1)\ , \qquad d^{-1}|0,k\rangle^j_{ba}=0$$

for $\mu_{ba}<0$, and

$$d^{-1}|n,k\rangle_{aa'}=\frac{1}{2\pi i\left(\left(\frac{n_1}{q_a+p_a\theta}-\beta_{aa'}\right)+\hat{\rho}\left(\frac{n_2}{q_a+p_a\theta}-\alpha_{aa'}\right)\right)}|n,k-1\rangle_{aa'}$$

for $\mu_{aa'}=0$. Here if $a=a'$, we set $d^{-1}|0,1\rangle=0$. Then we get the Hodge-Kodaira decomposition

$$dd^{-1}+d^{-1}d+P=\mathbf{1} \tag{67}$$

for each graded vector space V_{ab}. In particular, on V^0_{ab} one gets $d^{-1}d+P=\mathbf{1}$ since $d^{-1}=0$, and on V^1_{ab} one gets $dd^{-1}+P=\mathbf{1}$ since $d=0$.

3.2 Categorical mirror symmetry on noncommutative two-tori

We set up the homological mirror symmetry conjecture on noncommutative two-tori as an equivalence between two (cyclic) A_∞-categories. Just as a DG category is a natural extension of a DGA, an A_∞-category is a natural extension of an A_∞-algebra introduced by J. Stasheff [58, 59].

Definition 3.4 ((Cyclic) A_∞-category [16]) An A_∞-category $\mathcal{C}$ consists of a set of objects $\mathrm{Ob}(\mathcal{C})=\{a,b,\cdots\}$, $\mathbb{Z}$-graded vector space $\mathrm{Hom}(a,b)$ for each pair of objects $a,b\in\mathrm{Ob}(\mathcal{C})$, and a collection of degree one multilinear compositions $\mathfrak{m}:=\{m_n:\mathrm{Hom}(a_1,a_2)\otimes\mathrm{Hom}(a_2,a_3)\otimes\cdots\otimes\mathrm{Hom}(a_n,a_{n+1})\to\mathrm{Hom}(a_1,a_{n+1})\}_{a_i\in\mathrm{Ob}(\mathcal{C}),\ n\geq 1}$ satisfying the following relations

$$\begin{aligned} 0=&\sum_{k+l=n+1}\sum_{j=0}^{k-1}(-1)^{|e_{12}|+\cdots+|e_{j(j+1)}|} \\ &m_k(e_{12},\cdots,e_{j(j+1)},m_l(e_{(j+1)(j+2)},\cdots,e_{(j+l)(j+l+1)}),e_{(j+l+1)(j+l+2)},\cdots,e_{n(n+1)}) \end{aligned} \tag{68}$$

for $n \geq 1$. Here $|e_{i(i+1)}|$ on (-1) denotes the degree of $e_{i(i+1)} \in \mathrm{Hom}(a_i, a_{i+1})$. One can also express these relations as $0 = \sum_{k+l=n+1} \sum_{j=0}^{k-1} m_k(\mathbf{1}^{\otimes j} \otimes m_l \otimes \mathbf{1}^{\otimes (n-j-l)})(e_{12}, \cdots, e_{n(n+1)})$. Moreover, we call $\mathcal{C}$ a *cyclic A_∞-category* if it has a nondegenerate skew-symmetric inner product of fixed integer degree,

$$\omega : \mathrm{Hom}(a,b) \otimes \mathrm{Hom}(b,a) \to \mathbb{C} \ ,$$

satisfying the following cyclic condition

$$\omega(m_n(e_{12}, \cdots, e_{n(n+1)}), e_{(n+1)1}) = (-1)^{(|e_{23}|+\cdots+|e_{(n+1)1}|)|e_{12}|}\omega(m_n(e_{23}, \cdots, e_{(n+1)1}), e_{12}) \ .$$

For $\deg(\omega) = |\omega|$, the sign in the equation above can also be written as $(-1)^{(|e_{23}|+\cdots+|e_{(n+1)1}|)|e_{12}|} = (-1)^{(-|\omega|-1-|e_{12}|)|e_{12}|} = (-1)^{|\omega||e_{12}|}$.

Remark 3.5 An A_∞-category $\mathcal{C}$ consisting of one object only is equivalent to an A_∞-algebra [58, 59]. On the other hand, regarding $\mathcal{H} = \oplus_{a,b}\mathcal{H}_{ab}$, $\mathcal{H}_{ab} := \mathrm{Hom}(a,b)$ as a $\mathbb{Z}$-graded vector space, we can think of $(\mathcal{H}, \mathfrak{m})$ as an A_∞-algebra, and $(\mathcal{H}, \omega, \mathfrak{m})$ as a cyclic A_∞-algebra (for cyclic A_∞-algebras see [29]). Hereafter we sometimes denote a (cyclic) A_∞-category explicitly by $(\mathcal{H}, \mathfrak{m})$ or $(\mathcal{H}, \omega, \mathfrak{m})$.

Definition 3.5 (Minimal A_∞-category) An (cyclic) A_∞-category $\mathcal{C}$ is called *minimal* if $m_1 = 0$.

This corresponds to that $(\mathcal{H}, \mathfrak{m})$ with $m_1 = 0$ is a minimal A_∞-algebra.

Definition 3.6 ((Cyclic) A_∞-functor) For two A_∞-categories $\mathcal{C}$ and $\mathcal{C}'$, an A_∞-functor $\mathcal{F} : \mathcal{C} \to \mathcal{C}'$ consists of a map between the objects $\mathfrak{f} : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C}')$ and a collection of linear maps $\mathcal{F} := \{\mathfrak{f}_n : \mathrm{Hom}(a_1, a_2) \otimes \mathrm{Hom}(a_2, a_3) \otimes \cdots \otimes \mathrm{Hom}(a_n, a_{n+1}) \to \mathrm{Hom}(\mathfrak{f}(a_1), \mathfrak{f}(a_{n+1}))\}_{\substack{a_i \in \mathrm{Ob}(\mathcal{C}) \\ n \geq 1}}$ satisfying the following conditions

$$\sum_{k_1+\cdots+k_i=n} m'_i\,(\mathfrak{f}_{k_1} \otimes \cdots \otimes \mathfrak{f}_{k_i}) = \sum_{k+l=n+1} \sum_{j=0}^{k-1} \mathfrak{f}_k(\mathbf{1}^{\otimes j} \otimes m_l \otimes \mathbf{1}^{\otimes (n-j-l)}) \tag{69}$$

on $\mathrm{Hom}(a_1, a_2) \otimes \cdots \otimes \mathrm{Hom}(a_n, a_{n+1})$. Furthermore, for two cyclic A_∞-categories $\mathcal{C}$ and $\mathcal{C}'$ with their inner products ω and ω', respectively, we call an A_∞-functor $\mathcal{F} : \mathcal{C} \to \mathcal{C}'$ cyclic when

$$\omega'(\mathfrak{f}_1(e_{ab}), \mathfrak{f}_1(e_{ba})) = \omega(e_{ab}, e_{ba}) \ , \tag{70}$$

and for fixed $n \geq 3$,

$$\sum_{k,l \geq 1,\ k+l=n} \omega'(\mathfrak{f}_k(e_{12}, \cdots, e_{k(k+1)}), \mathfrak{f}_l(e_{(k+1)(k+2)}, \cdots, e_{n(n+1)})) = 0 \tag{71}$$

holds.

An A_∞-functor $\mathcal{F} : \mathcal{C} \to \mathcal{C}'$ induces an A_∞-morphism $\mathcal{F} : (\mathcal{H}, \mathfrak{m}) \to (\mathcal{H}', \mathfrak{m}')$, and a cyclic A_∞-functor $\mathcal{F} : \mathcal{C} \to \mathcal{C}'$ induces a cyclic A_∞-morphism $\mathcal{F} : (\mathcal{H}, \omega, \mathfrak{m}) \to (\mathcal{H}', \omega', \mathfrak{m}')$.

Definition 3.7 (Homotopy) An A_∞-functor $\mathcal{C} \to \mathcal{C}'$ is called *homotopy* if $\mathfrak{f} : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C}')$ is a bijection and $\mathfrak{f}_1 : \mathrm{Hom}(a,b) \to \mathrm{Hom}(\mathfrak{f}(a),\mathfrak{f}(b))$ induces an isomorphism on the cohomologies for each a and b. Also, it is called *fully faithful* if $\mathfrak{f}_1 : \mathrm{Hom}(a,b) \to \mathrm{Hom}(\mathfrak{f}(a),\mathfrak{f}(b))$ is an isomorphism. Alternatively, we call two cyclic A_∞-categories $\mathcal{C}$ and $\mathcal{C}'$ are homotopic if there exists a cyclic A_∞-functor $\mathcal{C} \to \mathcal{C}'$ which defines a homotopy as A_∞-categories.

A homotopy and a fully faithful A_∞-functor correspond to an A_∞-quasi-isomorphism and an A_∞-isomorphism, respectively, in the terminology of A_∞-algebras.

Definition 3.8 (Original definition) There exists another definition of an A_∞-category $\mathcal{C}$ which is related to the one in Definition 3.4 above. It consists of a set of objects $\mathrm{Ob}(\mathcal{C})$, a graded vector space V_{ab} for each objects $a, b \in \mathrm{Ob}(\mathcal{C})$ and a collection of multilinear maps $\mathfrak{m} := \{m_n : V_{a_1a_2} \otimes \cdots \otimes V_{a_na_{n+1}} \to V_{a_1a_{n+1}}\}$ of degree $(2-n)$ satisfying

$$\begin{aligned} 0 = &\sum_{k+l=n+1} \sum_{j=0}^{k-1} (-1)^\epsilon \\ &m_k(v_{12}, \cdots, v_{j(j+1)}, m_l(v_{(j+1)(j+2)}, \cdots, v_{(j+l)(j+l+1)}), v_{(j+l+1)(j+l+2)}, \cdots, v_{n(n+1)}) \ , \end{aligned} \tag{72}$$

where $\epsilon = (j+1)(l+1) + l(|v_{12}| + \cdots + |v_{j(j+1)}|)$. On this A_∞-category $\mathcal{C}$, a cyclic structure is given by a nondegenerate symmetric bilinear map $\eta : V_{ab} \otimes V_{ba} \to \mathbb{C}$ of fixed degree $|\eta| \in \mathbb{Z}$ satisfying

$$\eta(m_n(v_{12}, \cdots, v_{n(n+1)}), v_{(n+1)1}) = (-1)^{n+(|v_{23}|+\cdots+|v_{(n+1)1}|)|v_{12}|}\eta(m_n(v_{23}, \cdots, v_{(n+1)1}), v_{12}) \ , \tag{73}$$

for each $n \geq 1$.

Again, for $V := \oplus_{a,b} V_{ab}$, $(V, \mathfrak{m})$ forms an A_∞-algebra [58, 59], and $(V, \eta, \mathfrak{m})$ forms a cyclic A_∞-algebra (see [46]). As a cyclic DGA is an example of a cyclic A_∞-algebra, a cyclic DG category is an example of a cyclic A_∞-category in Definition 3.8 with $\eta = \eta$, $m_1 = d$, $m_2 = \varphi$ and $m_3 = m_4 = \cdots = 0$. The definition of a minimal A_∞-category is just the same, an A_∞-category with $m_1 = 0$. One can see that, for a minimal A_∞-category $\mathcal{C}$ of the original definition, the A_∞ relations (68) for $n = 3$ reduces to the associativity condition of the composition of morphisms m_2. Thus, $(\mathcal{H}, m_2)$ forms a category in a usual sense.

Anyway, both definitions of (cyclic) A_∞-categories are equivalent by the lemma below.

Lemma 3.2 For a graded vector space $V = \oplus_k V^k$ with k the grading, let

$$s : V^k \to V^{k-1}[1] =: \mathcal{H}^{k-1} \ , \qquad v_{ab} \mapsto e_{ab}$$

be a degree shifting operator called a suspension, where the indices $a, b, \cdots$ are preserved by this operation s. Then two cyclic A_∞-categories in Definition 3.4 and Definition 3.8 are compatible with each other through the suspension s.

proof. The equivalence of the two A_∞-categories is given in [19]. Let us distinguish the A_∞-structures in two notations by $\mathfrak{m}^{\mathcal{H}}$ and $\mathfrak{m}^V$. A relation between the multilinear maps is given by $m_n^{\mathcal{H}} = (-1)^{\sum_{i=1}^{n-1}(n-i)} s m_n^V((s^{-1})^{\otimes n})$ or more explicitly

$$\begin{aligned} m_n^{\mathcal{H}}(e_{12},\cdots,e_{n(n+1)}) &= (-1)^{\sum_{i=1}^{n-1}(n-i)e_{i(i+1)}} s m_n^V(s^{-1}(e_{12}),\cdots,s^{-1}(e_{n(n+1)})) \\ &= (-1)^{\sum_{i=1}^{n-1}(n-i)e_{i(i+1)}} s m_n^V(v_{12},\cdots,v_{n(n+1)}) \ . \end{aligned} \tag{74}$$

A relation between the two cyclic structures are also given by $\omega = \eta(s^{-1}\ ,s^{-1}\ )$, or

$$\omega(e,e') = (-1)^e \eta(s^{-1}(e), s^{-1}(e')) \ . \tag{75}$$

■

Remark 3.6 Through this relation, the definitions of minimal A_∞-categories (Definition 3.5), cyclic A_∞-functor (Definition 3.6), and homotopies (Definition 3.7) can be translated into those for cyclic A_∞-categories of Definition 3.8.

For an A_∞-category, the original definition (Definition 3.8) is natural from a homotopical point of view, where m_2 is a usual degree zero product, m_3 is a homotopy between the violation of associativity of m_2, and so on [58, 59]. However, Definition 3.4 is simpler in sign. Therefore, for a cyclic A_∞-category or a cyclic DG category, we take Definition 3.8 when we concentrate on the structure m_2 only, whereas in subsection 3.3 we use Definition 3.4 where higher multilinear maps are considered in general.

Definition 3.9 (Fukaya's A_∞-category on (non)commutative two-tori) For a fixed $\theta \in \mathbb{R}$, let us consider a two-torus $\hat{\mathbb{T}}^2$ whose covering space is $\widetilde{\hat{\mathbb{T}}^2} \simeq \mathbb{R}^2$ with coordinates $(x_1, x_2) \in \mathbb{R}^2$. We have $\pi_{12}\widetilde{\hat{\mathbb{T}}^2} = \hat{\mathbb{T}}^2$, $\pi_{12} := \pi_1\pi_2 = \pi_2\pi_1$, where π_1 and π_2 is the projections associated with the identifications $x_1 \sim x_1 + 1$ and $x_2 \sim x_2 + 1$, respectively. An object a is a geodesic cycle $\pi_{12}(L_a) \in \hat{\mathbb{T}}^2$,

$$L_a : q_a x_2 = p_a x_1 + \alpha_a \ , \qquad \alpha_a \in \mathbb{R} \ , \tag{76}$$

where p_a and q_a are relatively prime integers satisfying $q_a + p_a\theta > 0$, with a trivial line bundle equipped with a flat connection

$$\nabla_{a,1} = \frac{\partial}{\partial x_1} - 2\pi i \beta_a \ , \qquad \beta_a \in \mathbb{R} \ . \tag{77}$$

For each object a we assign a number

$$\mu_a := \frac{p_a}{q_a + p_a\theta} \ , \tag{78}$$

and for any two objects a and b, we define

$$\mu_{ab} := (\mu_b - \mu_a)(q_a + p_a\theta)^2 = \frac{p_{ab}}{q_{ab} + p_{ab}\theta_a} \ . \tag{79}$$

Note that by $SL(2,\mathbb{R})$ translation

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to \begin{pmatrix} x_1^\theta \\ x_2^\theta \end{pmatrix} = \begin{pmatrix} 1 & \theta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} ,$$

a line $x_1 + \theta x_2 = 0$ becomes a vertical line $x_1^\theta = 0$, μ_a is the slope of L_a after the translation, and μ_{ab} is $(q_a + p_a\theta)^2$ times the difference of the slope. The space of morphisms Hom is a graded vector space. We denote by k the grading and write $V_{ab}^F = \oplus_k V_{ab}^{F,k}$, $V_{ab}^{F,k} := \mathrm{Hom}^k(a,b)$. In $\hat{\mathbb{T}}^2$ there exists $|p_{ab}|$ (transversal) intersection points of $\pi_{12}(L_a)$ and $\pi_{12}(L_b)$ if $\mu_a \neq \mu_b$. These points are determined by projecting the intersection points of

$$L_a : q_a x_2 = p_a x_1 + \alpha_a \ , \qquad L'_b : q_b \left(x_2 - p_a \frac{q_{ab} j}{p_{ab}} \right) = p_b \left(x_1 - q_a \frac{q_{ab} j}{p_{ab}} \right) + \alpha_b$$

by $\pi_{12} : \widetilde{\hat{\mathbb{T}}^2} \to \hat{\mathbb{T}}^2$. We assign them by v_{ab}^j, $j = 0, 1, \cdots, |p_{ab}| - 1$. We identify $\{v_{ab}^j\}$ with the basis of V_{ab}^F. The grading of v_{ab}^j is 0 if $\mu_{ab} > 0$ and 1 if $\mu_{ab} < 0$. We denote the place of v_{ab}^j by $(x_1, x_2)(v_{ab}^j)$. Note that

$$(x_1, x_2)(v_{ba}^{-q_{ab} j}) = (x_1, x_2)(v_{ab}^j)$$

holds. In case $\mu_{ab} = 0$ (nontransversal case), L_a and L_b does not intersect with each other for $a \neq b$ and they coincide with each other if $a = b$. According to this, we formally introduce bases of morphisms $\mathbf{1}_a \in V_{aa}^{F,0}$ and $\bar{\mathbf{1}}_a \in V_{aa}^{F,1}$. They are uniformly denoted in general expression by v_{ab}^j with $a = b$ and $j = 0$; for $a = b$, if $v_{ab}^0 = \mathbf{1}_a$ then $v_{ba}^0 = \bar{\mathbf{1}}_a$, and if $v_{ab}^0 = \bar{\mathbf{1}}_a$ then $v_{ba}^0 = \mathbf{1}_a$. An A_∞ structure $\mathfrak{m}^F$ is defined by $m_1^F = 0$ and higher compositions m_n^F, $n \geq 2$ of degree $(2 - n)$, which are given by

$$\begin{aligned} m_n^F(v_{a_1 a_2}^{j_{12}}, \cdots, v_{a_n a_{n+1}}^{j_{n(n+1)}}) &= c_{a_1 \cdots a_{n+1}}^{j_{12} \cdots j_{(n+1)1}} \cdot v_{a_1 a_{n+1}}^{j_{1(n+1)}} \ , \\ c_{a_1 \cdots a_{n+1}}^{j_{12} \cdots j_{(n+1)1}} &= \sum_{\tilde{v} \in I_{a_1 \cdots a_{n+1}}^{j_{12} \cdots j_{(n+1)}}} sign(x_1^\theta(\tilde{v}_{a_1 a_2}^{j_{12}}) - x_1^\theta(\tilde{v}_{a_{n+1} a_1}^{j_{(n+1)1}})) \exp(2\pi i \hat{\rho} A(\tilde{v})) \exp(2\pi i \int \beta(\tilde{v})) \end{aligned} \tag{80}$$

if $\mu_{a_i a_{(i+1)}} \neq 0$ for any i. Here $I_{a_1 \cdots a_{n+1}}^{j_{12} \cdots j_{(n+1)1}}$ is the subset of

$$\{\tilde{v} = (\tilde{v}_{a_1 a_2}^{j_{12}}, \cdots, \tilde{v}_{a_n a_{n+1}}^{j_{n(n+1)}}, \tilde{v}_{a_{n+1} a_1}^{j_{(n+1)1}}) \in (\pi_{12}^{-1}(v_{a_1 a_2}^{j_{12}}), \cdots, \pi_{12}^{-1}(v_{a_n a_{n+1}}^{j_{n(n+1)}})), \pi_{12}^{-1}(v_{a_{n+1} a_1}^{j_{(n+1)1}})\}$$

satisfying the following conditions.

- $j_{(n+1)1} = -q_{a_{n+1} a_1} j_{1(n+1)}$.
- Interval $(\tilde{v}_{a_{i-1} a_i}^{j_{(i-1)i}}, \tilde{v}_{a_i a_{i+1}}^{j_{i(i+1)}})$ is included in $\pi_{12}^{-1}\pi_{12}(L_{a_i})$ for each $i = 1, \cdots, n+1$, where $\tilde{v}_{a_0 a_1}^{j_{01}} = \tilde{v}_{a_{n+1} a_1}^{j_{(n+1)1}} = \tilde{v}_{a_{n+1} a_{n+2}}^{j_{(n+1)(n+2)}}$.
- $\tilde{v}$ forms a convex.
- Modulo an equivalence relation; $\tilde{v}$ and $\tilde{v}'$ are identified with each other, if they coincides with a parallel transformation on $\widetilde{\hat{\mathbb{T}}^2}$. We can remove this equivalence relation by fixing a vertex, for instance, $\tilde{v}_{a_1 a_2}^{j_{12}} = v_{a_1 a_2}^{j_{12}} \in \widetilde{\hat{\mathbb{T}}^2}$ (more precisely, the image of $v_{a_1 a_2}^{j_{12}}$ by a natural inclusion $\hat{\mathbb{T}}^2 \hookrightarrow \widetilde{\hat{\mathbb{T}}^2}$).

$A(\tilde{v})$ is then the area of the convex $(n+1)$-gon, and $\int \beta(\tilde{v})$ is given by

$$\sum_{i=1}^{n+1}(x_1(\tilde{v}_{a_ia_{i+1}}^{j_{i(i+1)}}) - x_1(\tilde{v}_{a_{i-1}a_i}^{j_{(i-1)i}}))\beta_i \ , \tag{81}$$

where $x_1(\tilde{v}_{a_ia_{i+1}}^{j_{i(i+1)}})$ is the x_1-coordinate of the point $\tilde{v}_{a_ia_{i+1}}^{j_{i(i+1)}}$ on $\widetilde{\mathbb{T}^2}$. If at least one of $v_{a_ia_{i+1}}$, $i=1,\cdots,n+1$ is $\mathbf{1}_a$ in eq.(80), m_n^F vanishes for $n\geq 3$ and only when $n=2$, by regarding the corresponding area of the triangle is zero, we set $c_{a_1a_2a_3}^{j_{12}j_{23}j_{31}}=1$. This indicates that $\mathbf{1}_a$ for each a is the unit in this A_∞-category. Alternatively, if at least one of $v_{a_ia_{i+1}}$ is $\bar{\mathbf{1}}_a$, we set $c_{a_1\cdots a_{n+1}}^{j_{12}\cdots j_{(n+1)1}}=0$ for $n\geq 3$, and in case $n=2$ the rest two basis must be $\mathbf{1}_a$ so $c_{a_1a_2a_3}^{j_{12}j_{23}j_{31}}=1$. [4] One can define a degree minus one nondegenerate symmetric inner product $\eta^F : V_{ab}^F\otimes V_{ba}^F\to\mathbb{C}$ by

$$\eta^F(v_{ab}^j, v_{ba}^{j'}) = \delta_{[p_{ab}]}{}_{-q_{ab}j}^{j'} \ , \tag{82}$$

where by $\delta_{[p_{ab}]}$ we indicate the Kronecker's delta of mod $|p_{ab}|$, that is, $\delta_{[p_{ab}]}{}_j^{j'}=1$ when $(j'-j)\in p_{ab}\mathbb{Z}$ and zero otherwise. Then $(V^F,\eta^F,\mathfrak{m}^F)$ forms a cyclic A_∞-category in Definition 3.8. Since $m_1=0$, this is a minimal cyclic A_∞-category. One can see that $(V^F,\eta^F,\mathfrak{m}^F)$ is essentially independent of the choice of r_a and s_a. Though $q_{ab}=-s_ap_b+r_aq_b$, the ambiguity of the choice is absorbed into the permutations of $\{j=0,1,\cdots,|p_{ab}|-1\}$.

The fact that only convex $(n+1)$-gons are 'counted' is equivalent to the fact that $c_{a_1\cdots a_{n+1}}^{j_{12}\cdots j_{(n+1)1}}$ is nonzero only when $\sum_{i=1}^{n+1}\deg(v_{a_ia_{i+1}}^{j_{i(i+1)}})=-2+(n+1)$. The A_∞-relation follows from concentrating on a polygon which has one nonconvex vertex. There exist two ways to divide the polygon into two convex polygons. The corresponding terms then appear with opposite sign and cancel each other in the A_∞-relation.

Conjecture 3.1 (Homological mirror symmetry for noncommutative two-tori) Let $\mathcal{C}^A$ be a cyclic A_∞-category in Definition 3.4 given by Fukaya's A_∞-category $(V^F,\eta^F,\mathfrak{m}^F)$ in Definition 3.9 through the suspension in Lemma 3.2, and $\mathcal{C}^B$ a cyclic A_∞-category in Definition 3.4 given by the cyclic DG category (V,η,d,φ) in Proposition 3.1 through the suspension. Then, two cyclic A_∞-categories $\mathcal{C}^A$, $\mathcal{C}^B$ are homotopic to each other.

It is known that any A_∞-category is homotopic to a minimal A_∞-category. Such a homotopy algebraic property of A_∞-categories is quite the same as that of A_∞-algebras, for which this fact is shown by Kadeishvili [25] and called the minimal model theorem. A stronger version, decomposition theorem, also holds including the cyclic case [29]. Thus, a standard way to examine the equivalence of two cyclic A_∞-categories is to construct the corresponding minimal model of each cyclic A_∞-categories. In this situation, a homotopy between the two minimal models is, if there exists, a fully faithful cyclic A_∞-functor, which just corresponds to the ambiguity of minimal models. In our Conjecture 3.1, since $(V^F,\eta^F,\mathfrak{m}^F)$ is originally minimal,

[4] For the higher products m_n^F, $n\geq 3$ including $\bar{\mathbf{1}}_a$, the definition here is a tentative one. This in fact satisfies the A_∞-relation (72), but there exist other choices. These higher products including $\mathbf{1}_a$ or $\bar{\mathbf{1}}_a$ should be defined from Floer homology in the situation that some of lines are nontransversal (coincides with each other).

one may construct a minimal model of (V,η,d,φ). As presented later in subsection 3.3, there is a way to construct a minimal model [41, 36] based on the reduction on the cohomology space with respect to $d=m_1$. The minimal model of (V,η,d,φ) is then, by employing the Hodge-Kodaira decomposition $dd^{-1}+d^{-1}d+P=\mathbf{1}$ (Definition 3.3), a graded vector space PV equipped with inner product $\eta|_{PV}$, product $P\varphi$, and higher products, where $|_{PV}$ indicates the restriction onto PV. On the other hand, by forgetting the higher products of a minimal (cyclic) A_∞-category, one gets a (cyclic) graded category. If two cyclic A_∞-categories are homotopy equivalent, the corresponding two cyclic graded categories must be equivalent. Here we show this.

Theorem 3.1 On (non)commutative two-tori, a cyclic graded category (V^F,η^F,m_2^F) of the Fukaya's A_∞-category is equivalent to a cyclic graded category $(PV,\eta|_{PV},P\varphi)$ of holomorphic vector bundles. Namely, there exists an isomorphism $\mathfrak{f}_1: V_{ab}^{F,k}\to PV_{ab}^k$ for any a,b which is compatible with both the product structures and the cyclic structures

$$\mathfrak{f}_1 m_2^F = P\varphi(\mathfrak{f}_1\otimes\mathfrak{f}_1)\ ,\qquad \mathfrak{f}_1\eta^F=\eta|_{PV}(\mathfrak{f}_1\otimes\mathfrak{f}_1)\ . \tag{83}$$

proof. Since we already identify objects in both sides with each other and use the same notation $a,b,c,\cdots$, the map between the objects is just the identity map, For the morphisms, a map is defined by

$$\mathfrak{f}_1: v_{ab}^j\mapsto |0,0\rangle_{ab}^j \tag{84}$$

for degree zero morphisms and

$$\mathfrak{f}_1: v_{ab}^j\mapsto N_{ab}(0)|0,1\rangle_{ab}^j \tag{85}$$

for degree one morphisms. We shall denote the morphism corresponding to v_{ab}^j also by the same notation $v_{ab}^j\in PV$. The compatibility of $\mathfrak{f}_1$ with respect to the cyclicity in eq.(83) is easily checked directly. Now we show that the tensor product on the cohomology of the cyclic DG category $P\varphi$ coincides with the m_2^F in the Fukaya's A_∞-category side. The tensor product $\varphi(v_{ab}^{j_{ab}}\otimes v_{bc}^{j_{bc}})$ was computed in the case $\mu_{ab}>0$ and $\mu_{bc}>0$ in [12, 28, 47], where it was observed that the holomorphic vectors (elements of $H^0(V)$) close with respect to the tensor product. In general, the tensor product restricted on the cohomologies is given by

$$P\varphi(v_{ab}^{j_{ab}}\otimes v_{bc}^{j_{bc}})=\sum_{n=0}^{\infty}P|n,k\rangle_{ac}^j N_{ac}(n)\eta(\varphi(v_{ab}^{j_{ab}}\otimes v_{bc}^{j_{bc}}),|n,1-k\rangle_{ca}^{(-q_{ac}j)})\ ,$$

where k is the appropriate degree (zero or one). Since $P|n,k\rangle_{ac}^j=|0,k\rangle_{ac}^j$, it is expressed as

$$P\varphi(v_{ab}^{j_{ab}}\otimes v_{bc}^{j_{bc}})=c_{abc}^{j_{ab}j_{bc}(-q_{ac}j)}\cdot v_{ac}^j\ ,\qquad c_{abc}^{j_{ab}j_{bc}j_{ca}}:=\eta(\varphi(v_{ab}^{j_{ab}}\otimes v_{bc}^{j_{bc}}),v_{ca}^{j_{ca}})\ .$$

The definition of cyclic DG category guarantees that this structure constant $c_{abc}^{j_{ab}j_{bc}j_{ca}}\in\mathbb{C}$ is invariant with respect to the cyclic permutation $(a,b,c;j_{ab},j_{bc},j_{ca})\to(b,c,a;j_{bc},j_{ca},j_{ab})$. Therefore, to see the coincidence of this tensor product with the Fukaya category sides, it is enough

to check the coincidence of $c_{abc}^{j_{ab}j_{bc}j_{ca}}$ in both sides in the case $\deg(v_{ab}^{j_{ab}}) = \deg(v_{bc}^{j_{bc}}) = 0$, hence $\deg(v_{ca}^{j}) = 1$. There are the following four cases.

$$\begin{array}{lllll} \mu_{ab} & = 0, & = 0, & > 0, & > 0, \\ \mu_{bc} & = 0, & > 0, & = 0, & > 0, \\ \mu_{ca} & = 0, & < 0, & < 0, & < 0, \end{array} \quad . \tag{86}$$

The first three cases are trivial. When $c_{abc}^{j_{ab}j_{bc}j_{ca}}$ includes at least one element of $\mu = 0$, it reduces to the inner product and then $c_{abc}^{j_{ab}j_{bc}j_{ca}} = 1$. Let us compare $c_{abc}^{j_{ab}j_{bc}j_{ca}}$ in the last nontrivial case. For the tensor product side, by a direct calculation we get

$$c_{abc}^{j_{ab}j_{bc}j_{ca}} = \sum_{u\in\mathbb{Z}} \delta_{[p_{ab}]}{}^{j_{ab}}_{-r_{ab}u-r_{ac}j_{ca}} \delta_{[p_{bc}]}{}^{j_{bc}}_{u} \exp\left(\frac{2\pi i}{-p_{ab}p_{bc}p_{ca}}\left(\frac{\hat{\rho}}{2}(l_{abc}-\alpha_{abc})^2 - \beta_{abc}(l_{abc}-\alpha_{abc})\right)\right) , \tag{87}$$

$$l_{abc}(u, j_{ab}, j_{bc}, j_{ca}) := p_{ac}u + p_{bc}r_{ac}j_{ca} , \tag{88}$$

$$\alpha_{abc} := p_{ab}(q_c + p_c\theta)\alpha_{bc} - p_{bc}(q_a + p_a\theta)\alpha_{ab} = p_{bc}\alpha_a + p_{ca}\alpha_b + p_{ab}\alpha_c , \tag{89}$$

$$\beta_{abc} := p_{ab}(q_c + p_c\theta)\beta_{bc} - p_{bc}(q_a + p_a\theta)\beta_{ab} = p_{bc}\beta_a + p_{ca}\beta_b + p_{ab}\beta_c . \tag{90}$$

Here $\delta_{[p_{ab}]}$ and $\delta_{[p_{bc}]}$ are the Kronecker's delta of mod $|p_{ab}|$ and $|p_{bc}|$, respectively, defined in eq.(82).

On the other hand, in the A-model side (Definition 3.9), the A_∞-structure m_2 is given by the summation of triangles made from lines $\pi_{12}^{-1}\pi_{12}(L_a, L_b, L_c)$ on the covering space $\widetilde{\hat{\mathbb{T}}^2}$. Let us set the following lines in $\widetilde{\hat{\mathbb{T}}^2}$;

$$L_a : \ q_a x_2 = p_a x_1 + \alpha_a , \tag{91}$$

$$L_b' : \ q_b x_2 = p_b x_1 - q_{ab}j_{ab} + \alpha_b , \tag{92}$$

$$L_c' : \ q_c x_2 = p_c x_1 - q_{ac}j_{ab} - q_{bc}j_{bc} - p_{bc}m + \alpha_c . \tag{93}$$

L_b' is the slope p_b/q_b line which passes through

$$(x_1, x_2) = \left(q_a\frac{q_{ab}j_{ab}-\alpha_b}{p_{ab}}, p_a\frac{q_{ab}j_{ab}-\alpha_b}{p_{ab}}\right) ,$$

and L_c' is as the slope p_c/q_c line which passes through

$$(x_1, x_2) = \left(q_a\frac{q_{ac}j_{ab}}{p_{ac}} + q_b\left(m + \frac{q_{bc}j_{bc}-\alpha_c}{p_{bc}}\right), p_a\frac{q_{ac}j_{ab}}{p_{ac}} + p_b\left(m + \frac{q_{bc}j_{bc}-\alpha_c}{p_{bc}}\right)\right) .$$

We define $\tilde{v}_{ab}^{j_{ab}}$ (resp. $\tilde{v}_{bc}^{j_{bc}}$) by the intersection point of (L_a, L_b') (resp. (L_b', L_c')).

$$(x_1, x_2)(\tilde{v}_{ab}^{j_{ab}}) = \left(\frac{q_a q_{ab}j_{ab} - q_a\alpha_b + q_b\alpha_a}{p_{ab}}, \frac{p_a q_{ab}j_{ab} - p_a\alpha_b + p_b\alpha_a}{p_{ab}}\right) \tag{94}$$

$$(x_1, x_2)(\tilde{v}_{bc}^{j_{bc}}) = \left(\frac{q_b p_{bc}m + (q_b q_{ac} - q_c q_{ab})j_{ab} + q_b q_{bc}j_{bc} - q_b\alpha_c + q_c\alpha_b}{p_{bc}},\right. \tag{95}$$

$$\left.\frac{q_b p_{bc}m + (p_b q_{ac} - p_c q_{ab})j_{ab} + p_b q_{bc}j_{bc} - p_b\alpha_c + p_c\alpha_b}{p_{bc}}\right) . \tag{96}$$

Since the slopes of the three lines are given, the triangle is determined when the two points $v_{ab}^{j_{ab}}$ and $v_{bc}^{j_{bc}}$ are given. Namely, the triangle exists only when the intersection point of (L_a, L'_c)

$$(x_1, x_2) = \left(\frac{q_a(p_{bc}m + q_{ac}j_{ab} + q_{bc}j_{bc}) - q_a\alpha_c + q_c\alpha_a}{p_{ac}}, \frac{p_a p_{bc}m + p_a q_{ac}j_{ab} + p_a q_{bc}j_{bc} - p_a\alpha_c + p_c\alpha_a}{p_{ac}} \right)$$

coincides with

$$(x_1, x_2)(\tilde{v}_{ac}^{j_{ac}}) = \left(\frac{q_a q_{ac}j_{ac} - q_a\alpha_c + q_c\alpha_a}{p_{ac}}, \frac{p_a q_{ac}j_{ac} - p_a\alpha_c + p_c\alpha_a}{p_{ac}} \right) , \quad -q_{ac}j_{ac} = j_{ca} ,$$

which is obtained by replacing b, j_{bc} to c, j_{ac} in eq.(94). One can see that the condition is

$$p_{ab}m + r_{ca}j_{ab} + q_{bc}j_{bc} + j_{ca} \in p_{ca}\mathbb{Z} .$$

The structure constant is then given by

$$c_{abc}^{j_{ab}j_{bc}j_{ca}} = \sum_{m\in\mathbb{Z}} \delta_{[p_{ca}]}{}^{0}_{p_{bc}m + r_{ca}j_{ab} + q_{bc}j_{bc} + j_{ca}} \exp(2\pi i \hat{\rho} \triangle_m) \exp(2\pi i \int \beta_m) , \tag{97}$$

where $\triangle_m$ is the area of the triangle made from L_a, L'_b and L'_c which is computed as

$$\triangle_m = \frac{1}{-2p_{ab}p_{bc}p_{ca}} \left(l'_{abc}(m, j_{ab}, j_{bc}, j_{ca}) - \alpha_{abc} \right)^2 , \tag{98}$$

where α_{abc} is just that given in eq.(89) and

$$l'_{abc}(m, j_{ab}, j_{bc}, j_{ca}) := p_{ab}p_{bc}m + p_{ab}q_{bc}j_{bc} - p_{bc}j_{ab} . \tag{99}$$

The effect of the holonomy β is given by

$$\exp(2\pi i \int \beta_m) := \exp\left(-2\pi i \beta_{abc} \left(l'_{abc}(m, j_{ab}, j_{bc}, j_{ca}) - \alpha_{abc} \right)\right) , \tag{100}$$

where β_{abc} is also that given in eq.(90).

The rest of the proof is to show the coincidence between l_{abc} and l'_{abc} under the Kronecker's delta's in eq.(87) and eq.(97). First, $\delta_{[p_{ab}]}{}^{j_{ab}}_{-r_{ab}u - r_{ac}j_{ca}} = \delta_{[p_{ab}]}{}^{-q_{ab}j_{ab}}_{u + q_{ab}r_{ac}j_{ca}}$ can be deleted by putting

$$u = -q_{ab}j_{ab} - q_{ab}r_{ac}j_{ca} + p_{ab}(v + s_{ac}j_{ca}) = p_{ab}v - q_{ab}j_{ab} - r_{bc}j_{ca} \tag{101}$$

for $v \in \mathbb{Z}$. The term $s_{ac}j_{ca}$ is included for convenience. Substituting this u into l_{abc} one gets

$$l_{abc}(u = p_{ab}v - q_{ab}j_{ab} - r_{bc}j_{ca}) = -p_{ca}p_{ab}v + p_{ca}q_{ab}j_{ab} - p_{ab}j_{ca} . \tag{102}$$

On the other hand, combining eq.(101) with $\delta_{[p_{bc}]}{}^{j_{bc}}_{u}$ one gets a constraint for v as follows;

$$\begin{aligned} 0 =& q_{bc}\left(j_{bc} - (p_{ab}v - q_{ab}j_{ab} - r_{bc}j_{ca})\right) + p_{bc}m \\ =& p_{bc}m' + r_{ca}j_{ab} + q_{bc}j_{bc} + j_{ca} + p_{ca}v \end{aligned} \tag{103}$$

for $m \in \mathbb{Z}$ and $m' = m + r_{ab}v - s_{ab}j_{ab} - s_{bc}j_{ca}$. We denote by $I_{abc}^{j_{ab}j_{bc}j_{ca}}$ the set of v satisfies the constraint above;

$$I_{abc}^{j_{ab}j_{bc}j_{ca}} := \{v | \text{Exists } (v, m) \in \mathbb{Z}^2 s.t.\ 0 = p_{bc}m + r_{ca}j_{ab} + q_{bc}j_{bc} + j_{ca} + p_{ca}v\} \ . \tag{104}$$

Now, one can see that this constraint (104) is just the same as $\delta_{[p_{ca}]}{}^{0}_{p_{bc}m+r_{ca}j_{ab}+q_{bc}j_{bc}+j_{ca}}$ in eq.(97), and l'_{abc} in eq.(99) is equal to l_{abc} in eq.(102) under this constraint. Thus, we completed the coincidence of $c_{abc}^{j_{ab}j_{bc}j_{ca}}$ in (87) and (97).

Note that though the cyclicity of $c_{abc}^{j_{ab}j_{bc}j_{ca}}$ is guaranteed by the cyclicity of $\mathcal{V}$, now it can be seen explicitly. α_{abc} and β_{abc} are written just in manifest cyclic expression. Then l_{abc} in eq.(102) and l'_{abc} in eq.(99) are related by cyclic permutation of abc and $j_{ab}j_{bc}j_{ca}$ (with a flop $v \leftrightarrow m$). Moreover, instead of eq.(103), if one represent the constraint as

$$0 = (j_{bc} - (p_{ab}v - q_{ab}j_{ab} - r_{bc}j_{ca})) + p_{bc}m \ , \tag{105}$$

one can see that this is also related to eq.(103) by the cyclic permutation.

Note that by the definition of α_{abc} (89), β_{abc} (90) and the expression of $l_{abc}^{j_{ab}j_{bc}j_{ca}}$ in (102), one can see that the structure constant $c_{abc}^{j_{ab}j_{bc}j_{ca}}$ depend only on p_{ab}, q_{ab} and is independent of r_{ab}, s_{ab} in $\mathrm{Hom}(E_a, E_b)$. This fact justifies the correspondence of these two categories.

$\blacksquare$

For commutative two-tori, the categorical mirror symmetry is given by Polishchuk-Zaslow [48], where a map $\mathfrak{f}_1$ is given and shown to be compatible with the product structures on H^0 on both sides. Our result above, for $\theta = 0$, gives an explicit and stronger version of it in the sense that we showed it on all the cohomologies H^0 and H^1 together with the cyclicity.

3.3 Higher products associated to polygons

Recall that $\mathcal{C}^B = (\mathcal{H}, \omega^B, \mathfrak{m}^B)$ is a cyclic A_∞-category in Definition 3.4 given by the suspension (Lemma 3.2) of the cyclic DG category (V, d, φ, η) in Proposition 3.1. We relate V and $\mathcal{H}$ by the suspension s as

$$\begin{array}{cccc} s: & V_{ab}^k & \to & \mathcal{H}_{ab}^{k-1} \\ & |n,k\rangle_{ab}^j & \mapsto & |n,k-1\rangle_{ab}^j \ . \end{array} \tag{106}$$

The corresponding cyclic A_∞-category is given explicitly as $m_1^B = d$ and

$$\begin{aligned} m_2^B(|n,k\rangle_{ab}^j, |n',k'\rangle_{bc}^{j'}) &= (-1)^k \varphi(|n,k+1\rangle_{ab}^j, |n',k'+1\rangle_{bc}^{j'}) \ , \\ \omega^B(|n,k\rangle_{ab}^j, |n',k'\rangle_{ba}^{j'}) &= (-1)^k \eta(|n,k+1\rangle_{ab}^j, |n',k'+1\rangle_{ba}^{j'}) \ . \end{aligned} \tag{107}$$

We also define vertex map $\mathcal{V} : \mathcal{H}_{ab} \otimes \mathcal{H}_{bc} \otimes \mathcal{H}_{ca} \to \mathbb{C}$ by $\mathcal{V} := \omega^B(m_2^B \otimes \mathbf{1})$ for later convenience. We have

$$\mathcal{V}(|n,k\rangle_{ab}^j, |n',k'\rangle_{bc}^{j'}, |n'',k''\rangle_{ca}^{j''}) = (-1)^{k'+1}\eta(\varphi(|n,k+1\rangle_{ab}^j, |n',k'+1\rangle_{bc}^{j'}), |n'',k''+1\rangle_{ca}^{j''}) \ . \tag{108}$$

Note that, in the shifted degree, the dimension of $H^k(\mathcal{H}) := P\mathcal{H}$ can be nonzero only if k is equal to minus one or zero.

As stated previously, it is known that, for any A_∞-algebra, there exists a minimal A_∞-algebra which is A_∞-quasi-isomorphic to the original one [25]. An explicit way to construct a minimal A_∞-structure $\tilde{\mathfrak{m}}^B=\{\tilde{m}_2^B,\tilde{m}_3^B,\cdots\}$ is given for instance in [41, 36] based on homological perturbation theory (cf.[21, 22]). Since an A_∞-category can be thought of as an A_∞-algebra (Remark 3.5), the story is quite the same for an A_∞-category. We recall the way in concentrating on the case that the higher products of the original A_∞-structures are absent; $m_3^B=m_4^B=\cdots=0$. For some related literatures in physics, see [40, 62, 38, 39]. First, for a given Hodge-Kodaira decomposition $dd^{-1}+d^{-1}d+P=1$ for $d=m_1^B$, construct a correction of degree zero (degree preserving) multilinear maps $\mathfrak{f}_n: H(\mathcal{H}_{a_1a_2})\otimes\cdots\otimes H(\mathcal{H}_{a_na_{n+1}})\to\mathcal{H}_{a_1a_{n+1}}$ inductively by

$$\mathfrak{f}_n=\sum_{k,l\geq 1,\ k+l=n} d^{-1}m_2^B(\mathfrak{f}_k\otimes\mathfrak{f}_l) \tag{109}$$

with $\mathfrak{f}_1=\iota$ and $\mathfrak{f}_2=d^{-1}m_2^B$. The A_∞ structure on the cohomology is then given by degree one multilinear maps $\tilde{m}_n^B: H(\mathcal{H}_{a_1a_2})\otimes\cdots\otimes H(\mathcal{H}_{a_na_{n+1}})\to H(\mathcal{H}_{a_1a_{n+1}})$ as follows

$$\tilde{m}_n^B=\sum_{k,l\geq 1,\ k+l=n} Pm_2^B(\mathfrak{f}_k\otimes\mathfrak{f}_l) \tag{110}$$

for $\tilde{m}_2^B=Pm_2^B$. One can also express this construction in terms of (tree level) Feynman graphs (see [36, 27, 29]). Furthermore, when d^{-1} satisfies

$$\omega^B(d^{-1}\ ,\ )=\pm\omega^B(\ ,d^{-1}\ ) \tag{111}$$

the induced minimal A_∞-structure $\tilde{\mathfrak{m}}^B=\{\tilde{m}_n^B\}_{n\geq 2}$ has cyclicity with respect to $\omega|_{P\mathcal{H}}$, the inner product restricted onto the cohomology [27, 29]. The condition (111) holds if we take for d^{-1} the one given in Definition 3.3. Thus, $H(\mathcal{H},\omega,\mathfrak{m}^B):=(P\mathcal{H},\omega|_{PV},\tilde{\mathfrak{m}}^B)$ forms a minimal cyclic A_∞-category. In fact $\{\mathfrak{f}_n\}_{n\geq 1}$ is a cyclic A_∞-functor from the A_∞-category constructed above to the cyclic DG category in Lemma 3.1.

Now, let us consider a vertex $\mathcal{V}$

$$\mathcal{V}(f_{ab},f_{bc},f_{ca}):=\eta(m_2^B(f_{ab}\otimes f_{bc}),f_{ca}) \tag{112}$$

for $f_{ab}\in\mathcal{H}_{ab}$, $f_{bc}\in\mathcal{H}_{bc}$ and $f_{ca}\in\mathcal{H}_{ca}$. The product m_2 is written by inverting vertices as

$$m_2^B(f_{ab}\otimes f_{bc})=\sum_n |n,-1+k)_{ac}^j N_{ab}(n)\mathcal{V}(f_{ab},f_{bc},|n,-k)_{ca}^{-q_{ac}j})\ , \tag{113}$$

where $N_{ab}(n)\in\mathbb{R}$ is the normalization in eq.(66). We need only trivalent vertices one of the edges is the leaf of $H^0(\mathcal{H}_{ca})$ with $\mu_{ca}\leq 0$. The reason is as follows. The vertex $\mathcal{V}$ is related to m_2^B through eq.(113), and m_2^B connects to either d^{-1} as in eq.(109) or P as in (110). In the case of $d^{-1}m_2^B$, it does not vanish only if the image of m_2^B has degree zero. This implies $k=1$ in eq.(113), namely, the degree of elements which connect to internal edges are minus one. Elements of degree zero then contribute only to the leaves in terms of the tree graphs, that is, elements in $H^0(\mathcal{H})$. On the other hand, in eq.(113) when $k=1$, one of f_{ab},f_{bc} has degree zero

and another has degree minus one. Thus, for $d^{-1}m_2^B$ the corresponding vertex has an element in $H^0(\mathcal{H})$. As a result, eq.(109) can be reduced to

$$\mathfrak{f}_{n+1} = d^{-1}m_2^B(\mathfrak{f}_n \otimes \mathfrak{f}_1 + \mathfrak{f}_1 \otimes \mathfrak{f}_n) .$$

In the case of P acting on m_2^B in eq.(113), again the vertex contains a degree zero element. If it is f_{ab} or f_{bc}, the story is the same as above. If $|n,-k)_{ca}^{-q_{ac}j}$, only $|0,-k)_{ca}^{-q_{ac}j}$ survives because of the projection P.

Now we write down all the type of the vertices. By the cyclic permutation, let us set the elements of $H^0(\mathcal{H})$ as $|0,0)_{ca}^{j_{ca}}$ and consider vertex of the form

$$\mathcal{V}(|n,-1)_{ab}^{j_{ab}}, |m,-1)_{bc}^{j_{bc}}, |0,0)_{ca}^{j_{ca}}) . \tag{114}$$

There are eight types of vertices as follows.

$$\begin{array}{ccccccccc} & (000) & (><0) & (<>0) & (0><) & (>0<) & (>><) & (><<) & (<><) \\ \mu_{ab} & 0 & > & < & 0 & > & > & > & < \\ \mu_{bc} & 0 & < & > & > & 0 & > & < & > \\ \mu_{ca} & 0 & 0 & 0 & < & < & < & < & < \end{array} \tag{115}$$

where 0, $>$, $<$ indicate the sign of μ_{**}.

In the first three cases $((000), (>< 0), (<> 0))$, since $c = a$, the vertices just reduces to the inner product and we get

$$\mathcal{V}(|n,-1)_{ab}^{j_{ab}}, |m,-1)_{ba}^{j_{ba}}, |0,0)_a) = \frac{1}{N_{ab}(n)}\delta_{n,m} . \tag{116}$$

Next, let us consider the following type

$$\mathcal{V}(|n,-1)_{ab}^{j_{ab}}, |0,-1)_{bc}^{j_{bc}}, |0,0)_{ca}^{j_{ca}}) \tag{117}$$

for $\mu_{ab} > 0$ and $\mu_{bc} > 0$. This is also expressed by using the Hermite polynomial as $|n,-1)_{ab}^{j}$ is ;

$$\begin{aligned} \mathcal{V}(|n,-1)_{ab}^{j_{ab}}, |0,-1)_{bc}^{j_{bc}}, |0,0)_{ca}^{j_{ca}}) &= \frac{1}{N_{ac}(0))}\frac{(2\pi T(p_{abc}))^{\frac{n}{2}}}{(q_b+p_b\theta)^n(p_{ca})^n} \\ \sum_{v\in I_{abc}^{j_{ab}j_{bc}j_{ca}}} H_n\left(\sqrt{\frac{\pi T}{p_{abc}}}\bar{l}_{abc}^{j_{ab}j_{bc}j_{ca}}(v)\right) &\exp\left(\frac{2\pi i}{p_{abc}}\left(\frac{\hat{\rho}}{2}(\bar{l}_{abc}^{j_{ab}j_{bc}j_{ca}}(v))^2 - \beta_{abc}\bar{l}_{abc}^{j_{ab}j_{bc}j_{ca}}(v)\right)\right) , \end{aligned} \tag{118}$$

where we put $\bar{l}_{abc}^{j_{ab}j_{bc}j_{ca}}(v) := l_{abc}^{j_{ab}j_{bc}j_{ca}}(v) - \alpha_{abc}$ and $p_{abc} := -p_{ab}p_{bc}p_{ca}$ $(= -\mu_{ab}\mu_{bc}\mu_{ca})$. Since they are cyclic, except the coefficient, $H_n(\cdots)\exp(\cdots)$ in eq.(118) is invariant with respect to the cyclic permutation permutation $(a,b,c;j_{ab},j_{bc},j_{ca}) \to (b,c,a;j_{bc},j_{ca},j_{ab})$. The other types are obtained as follows. First, using the Leibniz rule (Lemma 2.3) we get $(>><)$ type vertices

$$\mathcal{V}(|n,-1)_{ab}^{j_{ab}}, |m,-1)_{bc}^{j_{bc}}, |0,0)_{ca}^{j_{ca}}) = (-1)^m\mathcal{V}(|n+m,-1)_{ab}^{j_{ab}}, |0,-1)_{bc}^{j_{bc}}, |0,0)_{ca}^{j_{ca}}) . \tag{119}$$

Here note that $\bar{\nabla}^\dagger|0,0)^{j_{ca}}_{ca}=0$. Next, replacing $\hat{\rho}$ with $\hat{\rho}^*$ in eq.(119) we get

$$\begin{aligned}&\mathcal{V}(|n,-1)^{j_{ab}}_{ab},|m,-1)^{j_{bc}}_{bc},|0,0)^{j_{ca}}_{ca})\\&=-\mathcal{V}(|n,-1)^{j_{ab}}_{ab},|m,0)^{j_{bc}}_{bc},|0,-1)^{j_{ca}}_{ca}) \qquad (120)\\&=\mathcal{V}(|n,0)^{j_{ab}}_{ab},|m,-1)^{j_{bc}}_{bc},|0,-1)^{j_{ca}}_{ca}) \qquad (121)\end{aligned}$$

for $\mu_{ab}<0$ and $\mu_{bc}<0$ (hence $\mu_{ca}>0$). More explicitly, by setting $m=0$ and permuting $(a,b,c;j_{ab},j_{bc},j_{ca})\to(b,c,a;j_{bc},j_{ca},j_{ab})$ in eq.(120) we obtain

$$\begin{aligned}\mathcal{V}(|0,-1)^{j_{ca}}_{ca},|n,-1)^{j_{ab}}_{ab},|0,0)^{j_{bc}}_{bc}) &= \frac{1}{N_{ba}(0)}\frac{(-2\pi T(p_{abc}))^{\frac{n}{2}}}{(q_c+p_c\theta)^n(p_{ab})^n}\\ \sum_{v\in I^{j_{ab}j_{bc}j_{ca}}_{abc}} H_n\left(\sqrt{\frac{-\pi T}{p_{abc}}}\bar{l}^{j_{ab}j_{bc}j_{ca}}_{abc}(v)\right)&\exp\left(\frac{2\pi i}{p_{abc}}\left(\frac{\hat{\rho}^*}{2}(\bar{l}^{j_{ab}j_{bc}j_{ca}}_{abc}(v))^2-\beta_{abc}\bar{l}^{j_{ab}j_{bc}j_{ca}}_{abc}(v)\right)\right),\end{aligned} \qquad (122)$$

and by setting $n=0$ and permuting $(a,b,c;j_{ab},j_{bc},j_{ca})\to(c,a,b;j_{ca},j_{ab},j_{bc})$ in eq.(121) we get a similar result

$$\begin{aligned}\mathcal{V}(|m,-1)^{j_{ab}}_{ab},|0,-1)^{j_{bc}}_{bc},|0,0)^{j_{ca}}_{ca}) &= \frac{1}{N_{cb}(0)}\frac{(-2\pi T(p_{abc}))^{\frac{n}{2}}}{(q_a+p_a\theta)^n(p_{bc})^n}\\ \sum_{v\in I^{j_{ab}j_{bc}j_{ca}}_{abc}} H_m\left(\sqrt{\frac{-\pi T}{p_{abc}}}\bar{l}^{j_{ab}j_{bc}j_{ca}}_{abc}(v)\right)&\exp\left(\frac{2\pi i}{p_{abc}}\left(\frac{\hat{\rho}^*}{2}(\bar{l}^{j_{ab}j_{bc}j_{ca}}_{abc}(v))^2-\beta_{abc}\bar{l}^{j_{ab}j_{bc}j_{ca}}_{abc}(v)\right)\right).\end{aligned} \qquad (123)$$

On the other hand, for the (><<) type vertex,

$$\mathcal{V}(|n,-1)^{j_{ab}}_{ab},|m,-1)^{j_{bc}}_{bc},|0,0)^{j_{ca}}_{ca})=\frac{m!}{(m-n)!}([\bar{\nabla}_{bc},\bar{\nabla}^\dagger_{bc}])^n\mathcal{V}(|0,-1)^{j_{ab}}_{ab},|m-n,-1)^{j_{bc}}_{bc},|0,0)^{j_{ca}}_{ca}) \qquad (124)$$

holds for $m\geq n$ and $\mathcal{V}(|n,-1)^{j_{ab}}_{ab},|m,-1)^{j_{bc}}_{bc},|0,0)^{j_{ca}}_{ca})=0$ for $m<n$ due to the Leibniz rule. Similarly, for the (<><) type vertex we have

$$\mathcal{V}(|n,-1)^{j_{ab}}_{ab},|m,-1)^{j_{bc}}_{bc},|0,0)^{j_{ca}}_{ca})=\frac{n!}{(n-m)!}(-[\bar{\nabla}_{ab},\bar{\nabla}^\dagger_{ab}])^m\mathcal{V}(|n-m,-1)^{j_{ab}}_{ab},|0,-1)^{j_{bc}}_{bc},|0,0)^{j_{ca}}_{ca}) \qquad (125)$$

for $n\geq m$ and $\mathcal{V}(|n,-1)^{j_{ab}}_{ab},|m,-1)^{j_{bc}}_{bc},|0,0)^{j_{ca}}_{ca})=0$ for $n<m$. The right hand sides of eq.(124) and eq.(125) are just given by eq.(120) and eq.(121), respectively.

The remaining vertices are those including $|n,k)^{j_{ab}}_{ab}$ with $\mu_{ab}=0$. They are also obtained by direct calculations. The (0 ><) type vertex is given by

$$\begin{aligned}\mathcal{V}(|n,-1)_{aa'},|0,-1)^{j_{a'b}}_{a'b},|0,0)^{j_{ba}}_{ba}) &= \frac{1}{N_{ab}(0)}\delta_{[p_{ab}]j_{ba}}^{-q_{ab}(j_{a'b}-n_2)}\exp\left(2\pi i n_1\frac{j_{a'b}q_{ab}}{p_{ab}}\right)\\ \exp\left(\frac{\pi(q_a+p_a\theta)(q_b+p_b\theta)}{p_{ab}}\right.&\left.\left(-\frac{1}{T}(n'_1+\hat{\rho}n'_2)^2+i\hat{\rho}(n'_2)^2+2i\left(\alpha_{ab}n'_1+\beta_{a'b}n'_2+\alpha_{ab}\beta_{aa'}\right)\right)\right),\end{aligned} \qquad (126)$$

where $n_1' := \frac{n_1}{q_a+p_a\theta} - 2\pi i\beta_{aa'}$ and $n_2' := \frac{n_2}{q_a+p_a\theta} - 2\pi i\alpha_{aa'}$. Similarly, the ($>0<$) type vertex is

$$\begin{aligned}&\mathcal{V}(|0,-1)_{ab}^{j_{ab}}, |n,-1)_{bb'}, |0,0)_{b'a}^{j_{b'a}}) = \frac{1}{N_{ab}(0)}\delta_{[p_{ab}]_{j_{b'a}}}^{-q_{ab}(j_{ab}-n_2r_{ab})}\exp\left(2\pi i n_1\left(\frac{j_{ab}}{p_{ab}} - n_2\theta_b\right)\right)\\ &\quad\exp\left(\frac{\pi(q_a+p_a\theta)(q_b+p_b\theta)}{p_{ab}}\left(-\frac{1}{T}(n_1'+\hat\rho n_2')^2 + i\hat\rho(n_2')^2 + 2i\left(\alpha_{ab'}n_1' + \beta_{ab}n_2' + \alpha_{ab'}\beta_{bb'}\right)\right)\right),\end{aligned} \tag{127}$$

where $n_1' := \frac{n_1}{q_b+p_b\theta} - 2\pi i\beta_{bb'}$ and $n_2' := \frac{n_2}{q_b+p_b\theta} - 2\pi i\alpha_{bb'}$.

We then get, due to the Leibniz rule,

$$\begin{aligned}\mathcal{V}(|n,-1)_{aa'}, |m,-1)_{a'b}^{j_{a'b}}, |0,0)_{ba}^{j_{ba}}) &= \frac{1}{(-(n_1'+\hat\rho n_2'))^m}\mathcal{V}(|n,-1)_{aa'}, |0,-1)_{a'b}^{j_{a'b}}, |0,0)_{ba}^{j_{ba}})\\ \mathcal{V}(|m,-1)_{ab}, |n,-1)_{bb'}^{j_{bb'}}, |0,0)_{b'a}^{j_{b'a}}) &= \frac{1}{(-(n_1'+\hat\rho n_2'))^m}\mathcal{V}(|0,-1)_{ab}, |n,-1)_{bb'}^{j_{bb'}}, |0,0)_{b'a}^{j_{b'a}}).\end{aligned} \tag{128}$$

Thus, we completed to determine all type of vertices and so the Feynman rules.

Lastly we end with giving an example of $\tilde m_3^B$ with

$$\mu_d < \mu_c < \mu_a < \mu_b\ . \tag{129}$$

In this case the basis of the cohomologies are $e_{ab}^{j_{ab}} = |0,-1)_{ab}^{j_{ab}} \in H^{-1}(\mathcal{H}_{ab})$, $e_{bc}^{j_{bc}} = N_{bc}(0)|0,0)_{bc}^{j_{bc}} \in H^0(\mathcal{H}_{bc})$, $e_{cd}^{j_{cd}} = N_{cd}(0)|0,0)_{cd}^{j_{cd}} \in H^0(\mathcal{H}_{cd})$ and $e_{ad}^{j_{ad}} = N_{ad}(0)|0,0)_{ad}^{j_{ad}} \in H^0(\mathcal{H}_{ad})$. Since $m_2^B(H^0\otimes H^0)$ vanishes by degree counting, we get

$$\begin{aligned}\tilde m_3^B(e_{ab}^{j_{ab}}, e_{bc}^{j_{bc}}, e_{cd}^{j_{cd}}) &= Pm_2(d^{-1}m_2(e_{ab}^{j_{ab}}, e_{bc}^{j_{bc}}), e_{cd}^{j_{cd}})\\ &= \mathcal{V}(e_{ab}^{j_{ab}}, e_{bc}^{j_{bc}}, |n,-1)_{ca}^{-q_{ac}j_{ac}})N_{ac}(n)\mathcal{V}(d^{-1}|n,0)_{ac}^{j_{ac}}, e_{cd}^{j_{cd}}, |0,-1)_{da}^{-q_{ad}j_{ad}})\cdot e_{ad}^{j_{ad}},\end{aligned} \tag{130}$$

where $d^{-1}|n,0)_{ac}^{j_{ac}} = |n-1,-1)_{ac}^{j_{ac}}$. By the Feynman rules obtained above, we have

$$\begin{aligned}&\mathcal{V}(e_{ab}^{j_{ab}}, e_{bc}^{j_{bc}}, |n,-1)_{ca}^{-q_{ac}j_{ac}}) = -\mathcal{V}(|n,-1)_{ca}^{-q_{ac}j_{ac}}, e_{ab}^{j_{ab}}, e_{bc}^{j_{bc}})\\ &\quad= -\frac{(2\pi T(p_{abc}))^{\frac{n}{2}}}{(q_a+p_a\theta)^n(p_{bc})^n}\sum_{v\in I_{abc}^{j_{ab}j_{bc}(-q_{ac}j_{ac})}} H_n\left(\sqrt{\frac{\pi T}{p_{abc}}}\bar l_{abc}^{j_{ab}j_{bc}(-q_{ac}j_{ac})}(v)\right)\\ &\qquad\qquad\cdot\exp\left(\frac{2\pi i}{p_{abc}}\left(\frac{\hat\rho}{2}(\bar l_{abc}^{j_{ab}j_{bc}(-q_{ac}j_{ac})}(v))^2 - \beta_{abc}\bar l_{abc}^{j_{ab}j_{bc}(-q_{ac}j_{ac})}(v)\right)\right),\\ &\mathcal{V}(|n-1,-1)_{ac}^{j_{ac}}, e_{cd}^{j_{cd}}, |0,-1)_{da}^{-q_{ad}j_{ad}}) = -\mathcal{V}(|n-1,-1)_{ac}^{j_{ac}}, N_{cd}(0)|0,-1)_{cd}^{j_{cd}}, |0,0)_{da}^{-q_{ad}j_{ad}})\\ &\quad= -\frac{N_{cd}(0)}{N_{ad}(0)}\frac{(-2\pi T(p_{acd}))^{\frac{n-1}{2}}}{(q_c+p_c\theta)^{n-1}(p_{da})^{n-1}}\sum_{w\in I_{acd}^{j_{ac}j_{cd}(-q_{ad}j_{ad})}} H_{n-1}\left(\sqrt{\frac{-\pi T}{p_{acd}}}\bar l_{acd}^{j_{ac}j_{cd}(-q_{ad}j_{ad})}(w)\right)\\ &\qquad\qquad\cdot\exp\left(\frac{2\pi i}{p_{acd}}\left(\frac{\hat\rho^*}{2}(\bar l_{acd}^{j_{ac}j_{cd}(-q_{ad}j_{ad})}(w))^2 - \beta_{acd}\bar l_{acd}^{j_{ac}j_{cd}(-q_{ad}j_{ad})}(w)\right)\right).\end{aligned} \tag{131}$$

Then we get

$$\tilde{m}_3^B(e_{ab}^{j_{ab}}, e_{bc}^{j_{bc}}, e_{cd}^{j_{cd}}) = \sum_{v \in I_{abc}^{j_{ab}j_{bc}(-q_{ac}j_{ac})}} \sum_{w \in I_{acd}^{j_{ac}j_{cd}(-q_{ad}j_{ad})}}$$
$$\exp\left(\frac{2\pi i}{p_{abc}}\left(\frac{\hat{\rho}}{2}(\bar{l}_{abc}^{j_{ab}j_{bc}(-q_{ac}j_{ac})}(v))^2 - \beta_{abc}\bar{l}_{abc}^{j_{ab}j_{bc}(-q_{ac}j_{ac})}(v)\right)\right)$$
$$\sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{p_{ab}p_{cd}}{p_{bc}p_{da}}\right)^{\frac{n}{2}} H_n\left(\sqrt{\frac{\pi T}{p_{abc}}}\bar{l}_{abc}^{j_{ab}j_{bc}(-q_{ac}j_{ac})}(v)\right) H_{n-1}\left(\sqrt{\frac{-\pi T}{p_{acd}}}\bar{l}_{acd}^{j_{ac}j_{cd}(-q_{ad}j_{ad})}(w)\right)$$
$$\exp\left(\frac{2\pi i}{p_{acd}}\left(\frac{\hat{\rho}^*}{2}(\bar{l}_{acd}^{j_{ac}j_{cd}(-q_{ad}j_{ad})}(w))^2 - \beta_{acd}\bar{l}_{acd}^{j_{ac}j_{cd}(-q_{ad}j_{ad})}(w)\right)\right) \tag{132}$$

From this one can observe that the structure constant is independent of θ as the structure constants of the Fukaya's A_∞-category (Definition 3.9) are.

Let us further consider

$$\tilde{m}_3^B(e_{ab}^0, e_{bc}^0, e_{cd}^0) \tag{133}$$

in the following situation $a = ((\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}), (\alpha_a, 0))$, $b = ((\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}), (\alpha_b, 0))$, $c = ((\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}), (0,0))$ and $d = ((\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}), (0,0))$. We have $g_{ab} = (\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix})$, $g_{bc} = (\begin{smallmatrix} 1 & -2 \\ 0 & 1 \end{smallmatrix})$, $g_{cd} = (\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix})$, $g_{ad} = (\begin{smallmatrix} 1 & -2 \\ 0 & 1 \end{smallmatrix})$ and $g_{ac} = (\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix})$. The triple product (132) then reduces to

$$\tilde{m}_3^B(e_{ab}^0, e_{bc}^0, e_{cd}^0) = \sum_{v \in 2\mathbb{Z}}\sum_{w \in \mathbb{Z}} \exp\left(\frac{\pi i \hat{\rho}}{2}(\bar{l}_{abc}^{000}(v))^2\right)$$
$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{2\pi}2^n n!}H_n\left(\sqrt{\frac{\pi T}{2}}\bar{l}_{abc}^{000}(v)\right) H_{n-1}\left(\sqrt{\frac{\pi T}{2}}\bar{l}_{acd}^{000}(w)\right)\exp\left(\frac{-\pi i \hat{\rho}^*}{2}(\bar{l}_{acd}^{000}(w))^2\right), \tag{134}$$

where $\bar{l}_{abc}^{000}(v) = -v + 2\alpha_a - \alpha_b$ and $\bar{l}_{acd}^{000}(w) = -2w + \alpha_a$. One can also rewrite this as

$$\sum_{\substack{s=v'-\alpha_a/2,\\ t=w'+\alpha_b/2\\ v',w' \in \mathbb{Z}}} \exp\left(2\pi i \hat{\rho}(2s+t)^2\right) \sum_{n=1}^{\infty} \frac{H_n\left(\sqrt{\frac{\pi T}{2}}(2s+t)\right) H_{n-1}\left(\sqrt{\frac{\pi T}{2}}s\right)}{\sqrt{2\pi}2^n n!} \exp\left(-2\pi i \hat{\rho}^*(s)^2\right)$$
$$= \sum_{\substack{s=v'-\alpha_a/2,\\ t=w'+\alpha_b/2\\ v',w' \in \mathbb{Z}}} \exp\left(2\pi i \hat{\rho}(3s^2+4st+t^2)\right) \sum_{n=1}^{\infty} \frac{H_n\left(\sqrt{\frac{\pi T}{2}}(2s+t)\right) H_{n-1}\left(\sqrt{\frac{\pi T}{2}}s\right)}{\sqrt{2\pi}2^n n!} \exp\left(-2\pi T(s)^2\right) \tag{135}$$

As in the example above, one can calculate all higher Massey products. This procedure is interesting since the results have some geometric interpretation in the A-model side. In the example above, one can see that $\tilde{m}_3^B$ is given by connecting two kind of triangles with the

propagator which is given by some summation of the Hermite polynomials. The value $\hat{\rho}(3s^2 + 4st + t^2))$ is then the area of the square used to define the A_∞-structure in the Fukaya category in Definition 3.9. However, the remaining part does not coincide with the one in the Fukaya category. These two A_∞-categories should be related by a homotopy, which is just a basis changing and is a nonlinear isomorphism in supermanifold description of A_∞-categories (see[27, 29]). We hope to construct it explicitly and complete the full homological mirror symmetry in a geometric way elsewhere.

4 Conclusions and Discussions

In this paper, we discussed a (homological) mirror symmetry on noncommutative two-tori. From a string theory viewpoint, we clarified the meaning of noncommutativity, especially, its relation to the D-brane stability, The homological mirror symmetry was then defined precisely, including the cyclic symmetry. We proved a part of it, the compatibility of product structures with the cyclicities, and gave a Feynman rules which calculates all the higher (Massey) products on $\mathcal{C}^B_\theta$ toward the explicit geometric proof of the homological mirror symmetry. Thus, the proof of the homological mirror symmetry for higher products has been left for a future direction. The problem is to construct an explicit homotopy between the minimal cyclic A_∞-categories.

On the other hand, expect for the higher products, we succeeded to have a deep understanding of a noncommutative mirror symmetry in the two-tori case. Thus, the next problem may be the extension of these results to higher (even) dimensional tori based on [26, 31, 55] together with results on the Morita equivalence for higher dimensional noncommutative tori (see [51, 52, 54, 44, 61]). Though seemingly straightforward, we can expect that it should includes fruitful structures, since in this case the noncommutativity should not be explained by the D-brane stability only. It should include the abelian variety case [17] and might be related to the lagrangian foliation [18]. Also, it is interesting if we could further extend and apply our arguments to homological mirror symmetry for torus fibrations [60].

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